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The ray tracing analytical solution within the RAMOD framework. The case of a Gaia-like observer

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Abstract

This paper presents the analytical solution of the inverse ray tracing problem for photons emitted by a star and collected by an observer located in the gravitational field of the Solar System. This solution has been conceived to suit the accuracy achievable by the ESA Gaia satellite (launched on 19 December 2013) consistently with the measurement protocol in General Relativity adopted within the RAMOD framework. The aim of this study is to provide a general relativistic tool for the science exploitation of such a revolutionary mission, whose main goal is to trace back star directions from within our local curved space-time, therefore providing a three-dimensional map of our Galaxy. The calculations are performed assuming that the massive bodies of the Solar System move uniformly and have monopole and quadrupole structures. The results are useful for a thorough comparison and cross-checking validation of what already exists in the field of relativistic astrometry. Moreover, the analytical solutions presented here can be extended to model other measurements that require the same order of accuracy as that expected for Gaia.

Keywords: light propagation, space astrometry, general relativity, gravitation

(Some figures may appear in colour only in the online journal)

1. Introduction

To fully exploit the science of the Gaia mission (ESA, [1]), a relativistic astrometric model needs to be able to cope with an accuracy of few micro-arcseconds (μas) for observations within the Solar System.

Gaia acts as a celestial compass, measuring arcs among stars with the purpose to determine their position via the absolute parallax method. The main goal is to construct a three-dimensional map of the Milky Way and unravel its structure, dynamics, and evolutional history. This task is accomplished through a complete census, to a given brightness limit, of about one billion individual stellar objects.

Since the satellite is positioned at Lagrangian point L2 of the Sun–Earth system, the measurements of Gaia are performed in a weak gravitational regime and the solution of Einstein's equation, i.e. the space-time metric, can take the general form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{1}$$

where $|h_{\alpha\beta}| \ll 1$ and $|\partial_i h_{\alpha\beta}| \ll 1$ can be treated as perturbations of a flat space-time, and represent all the Solar System contributions to the gravitational field. Their explicit expression, however, can be described in different ways according to the physical situation we are considering. This means that, for the weak-field case, $h_{\alpha\beta}$ can always be expanded in powers of a given smallness parameter, expansion usually made in powers of the gravitational constant G (post-Minkowskian approximation, PM) or of the quantity $1/c$ (post-Newtonian approximation, PN); note that in the PM approach one can still make a further expansion in powers of $1/c$ signifying that a solution for light rays in the PM approximation is more general than the corresponding solution in the PN one. The estimates performed inside the near-zone of the Solar System are sufficiently well supported by an approximation to the required order in the small parameter $\epsilon \equiv (v/c)$, which amounts to about 10^{-4} for the typical velocities of our planets. Moreover, for the propagation of the light inside the Solar System, the sources of gravity should be considered together with their internal structure and geometrical shape. This is particularly true when the light passes close to the giant planets. In other circumstances it is an unnecessary complication to consider the planets different from point-like objects especially when the model is devoted to the reconstruction of *stellar positions* in a global sense. However, at the μas level of accuracy, i.e. $(v/c)^3 = \epsilon^3$, the contribution to the metric coefficients by motion and internal structure of the giant planets needs to be taken into account, in particular if one wants to measure specific light deflection effects, as for example, those due to the quadrupolar terms. For this purpose, calculations are here performed assuming that the massive bodies of the Solar System move uniformly and have monopole and quadrupole structures.

The scope of this paper is to present an analytical solution for a null geodesic of the metric (1) consistently with the requirements of the Gaia astrometric mission and according to the RAMOD framework [2, 3]. RAMOD uses a 3 + 1 description of space-time in order to measure physical effects along the proper time and in the rest-space of a set of fiducial observers according to the following measurement protocol [4]:

(i) specify the phenomenon under investigation;
(ii) identify the covariant equations that describe the phenomenon;
(iii) identify the observer making the measurements;
(iv) choose a frame adapted to that observer allowing space-time splitting into the observer's space and time;

(v) understand the locality properties of the measurement under consideration (namely, whether it is local or non-local with respect to the background curvature);
(vi) identify the frame components of the quantities that are the observational targets;
(vii) find a physical interpretation of the above components following a suitable criterion;
(viii) verify the degree of the residual ambiguity, if any, in the interpretation of the measurements and decide the strategy to evaluate it (i.e., comparing to what already known).

The main procedure of the RAMOD approach is to express the null geodesic in terms of the physical quantities that enter the process of measurement, in order to entangle the entire light trajectory with the background geometry to the required approximations. Finally, the solution should be adapted to the relevant IAU resolutions considered for Gaia [5].

Solving the astrometric problem implies the compilation of an astrometric catalog with the accuracy of the measurement model. Indeed there exist several models conceived for the above task and formulated in different and independent ways ([6–8] and references therein). Their availability must not be considered as an 'oversized toolbox' provided by the theoretical physicists. Quite the contrary, they are needed to put future experimental results on solid grounds, especially if one needs to implement gravitational source velocities and retarded time effects. From the experimental point of view, in fact, modern space astrometry is well poised to cast the light of knowledge into a largely uncharted territory (see for example [9] in [10]). Such a huge push-forward will not only come from high-precision measurements, which demand suitable relativistic modeling, but also in the form of *absolute* results which need to be validated. In this regard, it is of capital importance to have *different* and *cross-checked* models, which can exploit different *solutions* to interpret the same experimental data.

For the reason above, inside the Consortium constituted for the Gaia data reduction (Gaia CU3, Core Processing, DPAC), two models were developed: (i) GREM (Gaia RElativistic Model, [6]) baselined for the Astrometric Global Iterative Solution (AGIS), and (ii) RAMOD (Relativistic Astrometric MODel) implemented in the Global Sphere Reconstruction (GSR) of the Astrometric Verification Unit at the Italian data center (DPCT, the only center, together with the DPC in Madrid, able to perform the calibration of positions, parallaxes and proper motions of the Gaia data). RAMOD was originated to satisfy the Gaia validation requirements; however, the procedure developed can, indeed, be extended to all physical measurements implying light propagation. It is the aim of this investigation to substantially improve the theory of light propagation within the RAMOD framework. In particular, in order to fully accomplish the precepts of the measurement protocol above, it is useful to isolate the contributions from the derivatives of the metric terms at the different orders retained. The choice of the space-time coordinates that justifies form (1) of the metric allows one to think that the metric perturbations and their derivatives mainly carry information about the background gravity.

The article is organized as follows. Section 2 is devoted to the definition of the mathematical environment needed to make the null geodesic explicit at the desired accuracy; to this purpose we introduce a suitable classification of the RAMOD equations in terms of the metric perturbations and their derivatives. In section 3, we set the appropriate approximations that allow the analytical solution of the astrometric problem. In section 4, we show the specific solution for light deflection by spherical and non-spherical gravitational sources. Finally, in section 5, we deduce the analytical solution of the trajectories of the light signal emitted by the stars and propagate through the gravitational field within the Solar System. In the last section we summarize the conclusions.

Below we list the notations used; we adopt signature +2.

Notations

- Greek indices run from 0 to 3, whereas Latin indices from 1 to 3;
- '$_{,\alpha}$': partial derivative with respect to the α coordinate;
- '$\cdot$': scalar product with respect to the Euclidean metric δ_{ij};
- '$\times$': cross product with respect to the Euclidean metric δ_{ij};
- tilde-ed symbols ' ~ ' refer to quantities related to the gravitational sources;
- repeated indices like $u_\alpha v^\alpha$ for any four vectors u^α, v^α means summation over their range of values;
- '$[\alpha\beta]$': antisymmetrization of the indices α, β;
- ∇: covariant derivative;
- ∇_α: α-component of the covariant derivative;
- $\nabla(f)$: spatial gradient of a function f;
- we use geometrized unit, namely $G = 1$ and $c = 1$.

2. The RAMOD equations for Gaia

The fundamental unknown of the RAMOD method is the space-like four-vector $\bar{\ell}^\alpha$, which is the projection of the tangent to the null geodesic onto the rest-space of the local barycentric observer, namely the one locally at rest with respect to the barycenter of the Solar System. Physically, such a four-vector identifies the *line-of-sight* of the incoming photon relative to that observer.

Once defined $\bar{\ell}^\alpha$, the equations of the null geodesic take a form which we shall refer to as the master-equations. Neglecting all the $O(h^2)$ terms, these read [3, 11]:

$$\frac{\mathrm{d}\bar{\ell}^0}{\mathrm{d}\sigma} - \bar{\ell}^i\bar{\ell}^j h_{0j,i} - \frac{1}{2}h_{00,0} = 0, \tag{2}$$

$$\begin{aligned}\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} &- \frac{1}{2}\bar{\ell}^k\bar{\ell}^i\left(\bar{\ell}^j h_{ij,0} - h_{00,i}\right) + \bar{\ell}^i\bar{\ell}^j\left(h_{kj,i} - \frac{1}{2}h_{ij,k}\right)\\ &+ \bar{\ell}^i\left(h_{k0,i} + h_{ki,0} - h_{0i,k}\right) - \frac{1}{2}h_{00,k} - \bar{\ell}^k\bar{\ell}^i h_{0i,0} + h_{k0,0} = 0.\end{aligned} \tag{3}$$

Here, σ is the affine parameter of the geodesic,

$$\mathrm{d}\sigma = \mathrm{d}t + \mathcal{O}(h), \tag{4}$$

and t is the coordinate time.

In order to solve the master equations one should define appropriate metric coefficients. To the order of ϵ^3 [1], which is what is required for the accuracy targeted for Gaia, one has to take into account the distance between the points on the photon trajectory and the barycenter of the ath gravity source at the appropriate retarded time, together with the dynamical contribution to the background metric by the relative motion of the gravitational sources[4]. More specifically $r^i_{(a)}$ is the retarded distance defined as

$$r^i_{(a)}\left(\sigma, \tilde{\sigma}'\right) = x^i(\sigma) - \tilde{x}^i_a\left(\tilde{\sigma}'\right), \tag{5}$$

[4] We want to stress that the effects arising from this retarded time are not those, aberration-like, of the light coming from a moving source, rather the deflections caused by a moving gravitational source on the photons coming from another light source.

where x^i and $\tilde{x}^i$ are the photon and source positions, and $\tilde{\sigma}'$ is the parameter considered as function of the time t' along the ath source world line. The retarded position of the source is fixed by the intercept of its worldline with the past light cone at any point on the photon trajectory. However, the retarded time $t' = t - r_{(a)}$ and the modulus of the retarded distance $r_{(a)}$ are intertwined in an implicit relation which would prevent us to solve the geodesic equations. Nonetheless, we show that it is possible to write an approximate form of the metric which retains the required order of accuracy of ϵ^3, but where the dependence on the retarded contribution is simplified.

By using the Taylor expansion around any $\tilde{\sigma}'$ to the first order in ϵ, we get for each source:

$$\tilde{x}^i_{(a)}(\tilde{\sigma}) \approx \tilde{x}^i_{(a)}\left(\tilde{\sigma}'\right) + \tilde{v}^i_{(a)}\left(\tilde{\sigma}'\right)\left(\tilde{\sigma} - \tilde{\sigma}'\right), \tag{6}$$

which allows to rewrite the retarded distance as

$$r^i_{(a)}\left(\sigma, \tilde{\sigma}'\right) \approx x^i(\sigma) - \tilde{x}^i_{(a)}(\tilde{\sigma}) + \tilde{v}^i_{(a)}\left(\tilde{\sigma}'\right)\left(\tilde{\sigma} - \tilde{\sigma}'\right),$$

i.e.

$$r^i_{(a)}\left(\sigma, \tilde{\sigma}'\right) = r^i_{(a)}(\sigma, \tilde{\sigma}) + \tilde{r}^i_{(a)}\left(\tilde{\sigma}, \tilde{\sigma}'\right) + O\left(\tilde{v}^2\right), \tag{7}$$

where we set $\tilde{r}^i_{(a)}(\tilde{\sigma}, \tilde{\sigma}') = \tilde{x}^i_{(a)}(\tilde{\sigma}) - \tilde{x}^i_{(a)}(\tilde{\sigma}')$. Nevertheless $(\tilde{\sigma}' - \tilde{\sigma})$ is again proportional to the retarded distance as measured along the source world line. In fact, considering the tangent four-vector of the source world line

$$\tilde{u}^\alpha_{(a)} = -\left(u_\beta \tilde{u}^\beta_{(a)}\right)\left(u^\alpha + \tilde{v}^\alpha_{(a)}\right),$$

where $\tilde{v}^\alpha_{(a)}$ is the α-component of the four-velocity of the source relative to the origin of the coordinate system and defined in the rest frame of the local barycentric observer $\mathbf{u}$[5], the interval elapsed from the position of the source at the time t' to that at t is

$$\begin{aligned}\tilde{\sigma}' - \tilde{\sigma} &= -\tilde{u}_{(a)\alpha}\Delta x^\alpha \\ &= -\eta_{\alpha\beta}\tilde{u}^\alpha_{(a)}\Delta x^\beta + O(h) \\ &\approx \Delta x^0 - \delta_{ij}\tilde{v}^i_{(a)}\Delta x^j + O(h),\end{aligned} \tag{8}$$

where $\Delta x^\alpha = x^\alpha(\tilde{\sigma}(t')) - x^\alpha(\tilde{\sigma}(t))$. To the first order in $\tilde{v}$, we have along the generator of the light cone

$$\Delta x^0 = r_{(a)}(\sigma, \tilde{\sigma}) + \frac{\left(\mathbf{r}_{(a)}(\sigma, \tilde{\sigma}) \cdot \tilde{\mathbf{r}}_{(a)}\left(\tilde{\sigma}, \tilde{\sigma}'\right)\right)}{r_{(a)}(\sigma, \tilde{\sigma})} + O\left(\tilde{v}^2\right), \tag{9}$$

then we get the following approximate expression for (5):

$$r^i_{(a)}\left(\sigma, \tilde{\sigma}'\right) \approx r^i_{(a)}(\sigma, \tilde{\sigma}) - \tilde{v}^i_{(a)}\left(\tilde{\sigma}'\right) r_{(a)}(\sigma, \tilde{\sigma}), \tag{10}$$

or

$$r^i_{(a)}\left(\sigma, \tilde{\sigma}'\right) \approx r_{(a)}(\sigma, \tilde{\sigma})\left[n^i_{(a)}(\sigma, \tilde{\sigma}) - \tilde{v}^i_{(a)}\left(\tilde{\sigma}'\right)\right], \tag{11}$$

where $n^i_{(a)}(\sigma, \tilde{\sigma}) = r^i_{(a)} / r_{(a)}$. This is equivalent, to first order in $\tilde{v}$, to the distance found in [12] and entering the expression of the metric, i.e.

[5] Any four-vector which is orthogonal to a time-like one is space-like and will occasionally be denoted as a spatial four-vector, see [4].

$$r_{(a)}\left(\sigma,\, \tilde{\sigma}'\right) \approx r_{(a)} - \mathbf{r}_{(a)} \cdot \tilde{\mathbf{v}}_{(a)}. \tag{12}$$

The choice for the perturbation term of the metric has to match the adopted retarded distance approximation together with the fact that the lowest order of the h terms is ϵ^2 and the present space astrometry accuracy does not exceed the ϵ^3 level.

2.1. The n-body spherical case

From the above discussion, it follows that a standard suitable solution of Einstein's equations in terms of a retarded tensor potential [3, 13], is

$$\begin{aligned} h_{00} &= \sum_a \frac{2\mathcal{M}_{(a)}}{r_{(a)}} + \mathcal{O}\left(\epsilon^4\right), \\ h_{0i} &= -\sum_a \frac{4\mathcal{M}_{(a)}}{r_{(a)}} \tilde{\beta}_{i\,(a)} + \mathcal{O}\left(\epsilon^5\right), \\ h_{ij} &= \sum_a \frac{2\mathcal{M}_{(a)}}{r_{(a)}} \delta_{ij} + \mathcal{O}\left(\epsilon^4\right), \end{aligned} \tag{13}$$

where $\mathcal{M}_{(a)}$ is the mass of the ath gravity source, $r_{(a)}$ is the absolute value of the position vector of the photon with respect to the source, $\tilde{\beta}^j = \tilde{x}^j_{,0} = (1 - h_{00}/2)\tilde{v}^i(\tilde{\sigma}) + O(h^2)$ is the coordinate spatial velocity of the gravity source. The above expression of the metric in case of monopoles can be further specialized according to the most convenient one suitable for each specific case, like the Liénard-Wiechert potentials used in [12].

Note that the time component of the tangent vector to the source's worldline [3] is

$$\tilde{u}^0 = \frac{\mathrm{d}t}{\mathrm{d}\tilde{\sigma}} = 1 + \frac{h_{00}}{2} + \frac{\tilde{v}^2}{2c^2} + O\left(h^2\right) + O\left(\epsilon^4\right) \tag{14}$$

while that of the local barycentric observer is

$$u^0 = \frac{\mathrm{d}t}{\mathrm{d}\sigma} = 1 + \frac{h_{00}}{2} + O\left(h^2\right). \tag{15}$$

Then from (14) and (15) we derive the following relationships in the linear approximation

$$\mathrm{d}\tilde{\sigma} = \mathrm{d}t\left(1 - \frac{h_{00}}{2} - \frac{\tilde{v}^2}{2}\right) + O\left(h^2\right) + O\left(\epsilon^4\right), \tag{16}$$

and

$$\mathrm{d}\sigma = \mathrm{d}t\left(1 - \frac{h_{00}}{2}\right) + O\left(h^2\right) + O\left(\epsilon^4\right). \tag{17}$$

Within the approximation (12) the perturbation of the metric is expressed as

$$h_{00} = 2\sum_a \frac{\mathcal{M}_{(a)}}{r_{(a)}}\left(1 + \mathbf{n}_{(a)}\cdot\tilde{\mathbf{v}}_{(a)}\right) + \mathcal{O}\left(\epsilon^4\right),$$

$$h_{0i} = -4\sum_a \frac{\mathcal{M}_{(a)}}{r_{(a)}} v_{i(a)} + \mathcal{O}\left(\epsilon^5\right),$$

$$h_{ij} = 2\sum_a \frac{\mathcal{M}_{(a)}}{r_{(a)}}\left(1 + \mathbf{n}_{(a)}\cdot\tilde{\mathbf{v}}_{(a)}\right)\delta_{ij} + \mathcal{O}\left(\epsilon^4\right), \tag{18}$$

or, by simplifying the notation

$$h_{00} = h \simeq \sum_a h_{(a)} = 2\sum_a \mathcal{M}_{(a)}\frac{1}{r_{(a)}}\left(1 + \mathbf{n}_{(a)}\cdot\tilde{\mathbf{v}}_{(a)}\right) + \mathcal{O}\left(\epsilon^4\right),$$

$$h_{0i} = -2\,h\tilde{v}_i \simeq -2\sum_a h_{(a)}\tilde{v}_{i(a)} + \mathcal{O}\left(\epsilon^3\right),$$

$$h_{ij} \simeq h\,\delta_{ij} + \mathcal{O}\left(\epsilon^4\right). \tag{19}$$

In the following, $r_{(a)}$, unless explicitly expressed, will indicate a function with arguments σ and $\tilde{\sigma}$. Moreover, to ease notation we drop the index (a) wherever it is not necessary.

Let us consider a space-time splitting with respect to the congruence of fiducial observers **u** in the gravitational field of the Solar System [2]. The field equations can be rewritten in terms of the shear, expansion and vorticity of the congruence **u** (see [13]). For our purpose it is enough to consider only the expansion term and the vorticity (see [11]).

The master equations (2) and (3) are obtained by retaining the vorticity term at least to the order of $O(h_{0i})$, and the expansion to the order of $O(\partial_0 h_{00})$ and $O(\partial_0 h_{0i})$. In the case of a vorticity and expansion-free geometry, the RAMOD master equations are named RAMOD3 master equations [2]

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} + \bar{\ell}^i\bar{\ell}^j\left(h_{kj,i} - \frac{1}{2}h_{ij,k}\right) + \frac{1}{2}\bar{\ell}^k\bar{\ell}^i h_{00,i} - \frac{1}{2}h_{00,k} + \mathcal{O}\left(h^2\right) = 0, \tag{20}$$

where $\bar{\ell}^0 = 0$. Taking into account that $\bar{\ell}^i\bar{\ell}^j\delta_{ij} = 1 + \mathcal{O}\left(\epsilon^2\right)$, equations (2), (3), and (20) can be reduced respectively to:

$$\frac{\mathrm{d}\bar{\ell}^0}{\mathrm{d}\sigma} = -2\left(\bar{\ell}\cdot\tilde{v}\right)\left(\bar{\ell}^i h_{,i}\right) + \frac{1}{2}h_{,0} + \mathcal{O}\left(h^2\right), \tag{21}$$

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} = -\frac{3}{2}\bar{\ell}^k\left(\bar{\ell}^i h_{,i}\right) + h_{,k} - \frac{1}{2}\bar{\ell}^k h_{,0} - 2\left(\bar{\ell}\cdot\tilde{v}\right)h_{,k} + 2\tilde{v}^k\left(\bar{\ell}^i h_{,i}\right) - \bar{\ell}^k\bar{\ell}^i\left(\tilde{v}_i h\right)_{,0} \tag{22}$$

$$+\left(\tilde{v}^k h\right)_{,0} + \mathcal{O}\left(h^2\right) \tag{23}$$

and for the static case ($h_{\alpha\beta,0} = 0$ and $\tilde{v} = 0$)

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} + \frac{3}{2}\bar{\ell}^k\left(\bar{\ell}^i h_{,i}\right) - h_{,k} + \mathcal{O}\left(h^2\right) = 0. \tag{24}$$

As mentioned in the introduction, in order to fully accomplish the precepts of the measurement protocol, it is useful to isolate the contributions from the derivatives of the metric at the different retained orders.

This allows us to classify the master equation as follows:

- RAMOD3a (R3a), the spatial derivatives of the metric are considered while h_{0i} are neglected

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} = -\frac{3}{2}\bar{\ell}^k\left(\bar{\ell}^i h_{,i}\right) + h_{,k} + \mathcal{O}\left(h^2\right) \equiv (\mathrm{R3a})^k. \tag{25}$$

- RAMOD3b (R3b), the spatial and time derivatives of the metric are considered while h_{0i} are neglected

$$\frac{\mathrm{d}\bar{\ell}^0}{\mathrm{d}\sigma} = \frac{1}{2}h_{,0} + \mathcal{O}\left(h^2\right) \equiv (\mathrm{R3b})^0, \tag{26}$$

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} = (\mathrm{R3a})^k - \frac{1}{2}\bar{\ell}^k h_{,0} + \mathcal{O}\left(h^2\right) \equiv (\mathrm{R3b})^k. \tag{27}$$

- RAMOD4a (R4a), the spatial derivatives of the metric are considered including h_{0i}

$$\frac{\mathrm{d}\bar{\ell}^0}{\mathrm{d}\sigma} = (\mathrm{R3b})^0 - 2\left(\bar{\ell}\cdot\tilde{v}\right)\left(\bar{\ell}^i h_{,i}\right) + \mathcal{O}\left(h^2\right) \equiv (\mathrm{R4a})^0, \tag{28}$$

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} = (\mathrm{R3b})^k - 2\left(\bar{\ell}\cdot\tilde{v}\right)h_{,k} + 2\tilde{v}^k\left(\bar{\ell}^i h_{,i}\right) + \mathcal{O}\left(h^2\right) \equiv (\mathrm{R4a})^k. \tag{29}$$

- RAMOD4b (R4b), the spatial and time derivatives of the metric are considered including h_{0i}

$$\frac{\mathrm{d}\bar{\ell}^0}{\mathrm{d}\sigma} = (\mathrm{R4a})^0 + \mathcal{O}\left(h^2\right) \equiv (\mathrm{R4b})^0, \tag{30}$$

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} = (\mathrm{R4a})^k - 2\bar{\ell}^k\bar{\ell}^i\left(\tilde{v}_i h\right)_{,0} + 2\left(\tilde{v}^k h\right)_{,0} + \mathcal{O}\left(h^2\right) \equiv (\mathrm{R4b})^k. \tag{31}$$

Given the nature of the expansions, we expect that the solutions will reflect consistently the order of accuracy of the different classes; however it is certainly possible that their ranges can overlap in different geometric configurations between the sources and the observer. Moreover, the classification that we have so far introduced turns out to be extremely useful for the implementation of RAMOD models and the testing of them through consistent internal checks at different levels of accuracy, allowing also a very simple procedure to identify where the possible discrepancies can arise.

Beside this new classification of the RAMOD master equations, it is clear that the solutions call for an explicit expression of the metric terms. In general, for any integer m:

$$(r)^m_{,i} = m r^i (r)^{m-2}, \quad (r)^m_{,0} = -m\left(\tilde{\beta}\cdot r\right)(r)^{m-2}. \tag{32}$$

From the last computation one could expect that in the case of mapped trajectories for a RAMOD3-like model (see [2, 11]) the term $x^i_{,0}$ should be retained, since each mapped spatial coordinate depends on the σ value of the local one-parameter diffeormorphism. In this respect, note that the null geodesic crosses each slice $S(t)$ at a point with coordinates $x^i(\sigma(t))$, but this point also belongs to the unique normal to the slice $S(t)$ crossing it with a value $\sigma = \sigma(x^i, t)$

that runs differently for any spatial coordinate and therefore does not coincide with the proper time of the local barycentric observer. Therefore

$$\partial_0 x^i = \left(\partial_\sigma x^i(\sigma)\right)\left(\partial_0 \sigma\left(x^i(\lambda),\, t\right)\right) = 0.$$

Now, by using the retarded time approximation we get:

$$(r)^{-m}_{,i}\left(\sigma,\, \tilde{\sigma}'\right) = -m r_{(a)}^{-m-1}\left(1 + n_{(a)} \cdot \tilde{v}\right)^{m-1}\left[n^i_{(a)}\left(1 + 2n_{(a)} \cdot \tilde{v}\right) - \tilde{v}^i\right] + O(h) \tag{33}$$

and

$$(r)^{-m}_{,0}\left(\sigma,\, \tilde{\sigma}'\right) = m r_{(a)}^{-m-1}\left(n_{(a)} \cdot \tilde{\beta}\right)\left(1 + n_{(a)} \cdot \tilde{\beta}\right)^m + O(h). \tag{34}$$

Let us define the photon impact parameter with respect to the gravity source position as:

$$d^k_{(a)} = r^k_{(a)} - \bar{\ell}^k\left(r_{(a)} \cdot \bar{\ell}\right) \tag{35}$$

and, for sake of convenience, let us define also:

$$d^k_{v_{(a)}} = \tilde{v}^k_{(a)} - \bar{\ell}^k\left(\tilde{v}_{(a)} \cdot \bar{\ell}\right). \tag{36}$$

Finally, according to the previous derivatives, making them explicit, denoting $n^k_{(a)} = r^k_{(a)} \big/ r_{(a)}$ and $d^k_{(a)} \big/ r_{(a)} = d^k_{n_{(a)}}$, the master equations assume the following expressions, valid up to the ϵ^3 order:

- RAMOD3a:

$$\begin{aligned}\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} = 2\sum_a \frac{\mathcal{M}_{(a)}}{r^2_{(a)}}\Bigg\{&\left(\frac{1}{2}\bar{\ell}^k\left(n_{(a)} \cdot \bar{\ell}\right) - d^k_{n_{(a)}}\right)\left(1 + 2n_{(a)} \cdot \tilde{v}_{(a)}\right)\\ &- \frac{1}{2}\bar{\ell}^k\left(\bar{\ell} \cdot \tilde{v}_{(a)}\right) + d^k_{v_{(a)}}\Bigg\} + O\left(\tilde{v}^2\right) + O\left(h^2\right),\end{aligned} \tag{37}$$

where in case of zero velocity we recover the static RAMOD recorded as follows

- RAMOD3s (R3s):

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} = 2\sum_a \frac{\mathcal{M}_{(a)}}{r^2_{(a)}}\left\{\frac{1}{2}\bar{\ell}^k\left(n_{(a)} \cdot \bar{\ell}\right) - d^k_{n_{(a)}}\right\} + O\left(h^2\right). \tag{38}$$

Similarly for the other classification items we have:

- RAMOD3b:

$$\frac{\mathrm{d}\bar{\ell}^0}{\mathrm{d}\sigma} = \sum_a \frac{\mathcal{M}_{(a)}}{r^2_{(a)}}\left(n_{(a)} \cdot \tilde{v}\right) + O\left(\tilde{v}^2\right) + O\left(h^2\right), \tag{39}$$

$$\frac{\mathrm{d}\bar{\ell}^k}{\mathrm{d}\sigma} = (\mathrm{R3a})^k - \bar{\ell}^k \sum_a \frac{\mathcal{M}_{(a)}}{r^2_{(a)}}\left(n_{(a)} \cdot \tilde{v}\right) + O\left(\tilde{v}^2\right) + O\left(h^2\right). \tag{40}$$

- RAMOD4a

$$\frac{\mathrm{d}\bar{\ell}^0}{\mathrm{d}\sigma} = (\mathrm{R3b})^0 + 4\sum_a \frac{\mathcal{M}_{(a)}}{r^2_{(a)}}\left[\left(\bar{\ell} \cdot \tilde{v}\right)\left(\bar{\ell} \cdot n_{(a)}\right)\right] + O\left(\tilde{v}^2\right) + O\left(h^2\right), \tag{41}$$

$$\frac{d\bar{\ell}^k}{d\sigma} = (\mathrm{R3b})^k \quad + \quad 4\sum_a \frac{\mathcal{M}_{(a)}}{r_{(a)}^2}\left[\bar{\ell} \times \left(n_{(a)} \times \tilde{v}\right)\right]^k + O\left(\tilde{v}^2\right) + O\left(h^2\right). \tag{42}$$

It is clear that, to the order of ϵ^3, we do not need to include the time derivative of h_{0k} since these are at least of the order of ϵ^4 and should be neglected. Therefore, we do not consider RAMOD4b, as defined earlier, which contains the time derivative of h_{0k} and the second order velocity contributions.

2.2. The case for an oblate body

Now, let us consider the ath source and define

$$h_{(a)} = 2M_{(a)}\bar{h}_{(a)} \equiv 2M_{(a)}\frac{1}{r_{(a)}}\left[1 - \sum_{m=2}^{\infty} J_m \left(\frac{R_{(a)}}{r_{(a)}}\right)^m P_m\left(\cos\theta_{(a)}\right)\right], \tag{43}$$

which means to take into account the mass multipole structure of the ath body where P_m are the Legendre polynomials, $M_{(a)}$ the mass of the body, $R_{(a)}$ its equatorial radius, $\theta_{(a)}$ the co-latitude, and J_m the coefficients of the mass multipole moments. With this choice our considerations are confined to the case in which the object ellipsoid of inertia is an ellipsoid of revolution and the directions of the spatial coordinate axes coincide with those of the principal axes of inertia [14].

A rigorous treatment of an n-body multipolar expansion should take into account the different orientations of its axis of symmetry. However, this contribution decreases so quickly that at any accuracy currently attainable it turns out to be an unnecessary complication, since just one planet at a time would give a detectable effect.

Considering equations (12) and (19), the derivatives of the metric coefficients, with retarded time approximation, have the following expressions:

$$\begin{aligned}
\bar{h}_{(a),k} = &\left(-\frac{n_{(a)}^k}{r_{(a)}^2}\left(1 + 2n_{(a)}\cdot\tilde{v}\right) + \frac{\tilde{v}^k}{r_{(a)}^2}\right)\left[1 - \sum_{m=2}^{\infty} J_m\left(\frac{R_{(a)}}{r_{(a)}}\right)^m P_m\left(\cos\theta_{(a)}\right)\right] \\
&+ \frac{1}{r_{(a)}}\left(1 + n_{(a)}\cdot\tilde{v}\right)\left\{\sum_{m=2}^{\infty} J_m\left(\frac{R_{(a)}}{r_{(a)}}\right)^m \left[m\left(1 + n_{(a)}\cdot\tilde{v}\right)^m P_m\left(\cos\theta_{(a)}\right)\right.\right. \\
&\left.\left.\left(\frac{n_{(a)}^k}{r_{(a)}} - \frac{\tilde{v}^k}{r_{(a)}}\left(1 + \delta_{ij}n_{(a)}^i\tilde{v}^j\right)^{-1}\right) - P_m\left(\cos\theta_{(a)}\right)_{,k}\right]\right\},
\end{aligned} \tag{44}$$

$$\begin{aligned}
\bar{h}_{(a),0} = &\left(\frac{n_{(a)}\cdot\tilde{\beta}}{r_{(a)}^2}\right)\left(1 + n_{(a)}\cdot\tilde{\beta}_{(a)}\right)\left[1 - \sum_{m=2}^{\infty} J_m\left(\frac{R_{(a)}}{r_{(a)}}\right)^m P_m\left(\cos\theta_{(a)}\right)\right] \\
&+ \frac{1}{r_{(a)}}\left(1 + n_{(a)}\cdot\tilde{v}\right)\left[-\sum_{m=2}^{\infty} J_m\left(\frac{R_{(a)}}{r_{(a)}}\right)^m \left(P_m\left(\cos\theta_{(a)}\right)_{,0}\right.\right. \\
&\left.\left.+ mP_m\left(\cos\theta_{(a)}\right)\frac{n_{(a)}\cdot\tilde{\beta}}{r_{(a)}^2}\left(1 + n_{(a)}\cdot\tilde{v}\right)^m\right)\right],
\end{aligned} \tag{45}$$

where $n^i = r^i/r$. A general n-body solution should include the multipolar structure of the sources. Nevertheless, according to the current astrometric accuracy and for an oblate body, the quadrupole approximation can be considered enough (see [6]). If we omit the higher multipole moments and restrict ourselves only to $m = 2$, denoting by s^k the axis of the sources which is normal to the source equatorial plane, $P_m(\cos\theta_{(a)})$ is approximated as:

$$P_2\left(\cos\theta_{(a)}\right) = \frac{3\left(s_{(a)}\cdot r\left(\sigma, \tilde{\sigma}'\right)\right)^2}{2r\left(\sigma, \tilde{\sigma}'\right)^2} - \frac{1}{2} \approx \frac{3}{2}\left(s_{(a)}\cdot n_{(a)} - s_{(a)}\cdot\tilde{v}\right)^2\left(1 + n_{(a)}\cdot\tilde{v}\right)^2 - \frac{1}{2}, \tag{46}$$

which to first order in $\tilde{v}^i$ becomes

$$P_2\left(\cos\theta_{(a)}\right) = \frac{3}{2}(s_{(a)}\cdot n_{(a)})^2\left(1 + 2n_{(a)}\cdot\tilde{v}\right) - 3(s_{(a)}\cdot n_{(a)})\left(s_{(a)}\cdot\tilde{v}\right) - \frac{1}{2} + O\left(\tilde{v}^2\right); \tag{47}$$

therefore

$$\begin{aligned}\bar{h}_{(a),k} = &-\frac{n^k_{(a)}}{r^2_{(a)}}\left(1 + 2n_{(a)}\cdot\tilde{v}\right) + \frac{\tilde{v}^k}{r^2_{(a)}} + J_2 R^2_{(a)}\left\{\frac{3n^k_{(a)}}{r^4_{(a)}}\left[\frac{5(s_{(a)}\cdot n_{(a)})^2}{2}\right.\right.\\ &\left.+\ 9(s_{(a)}\cdot n_{(a)})^2\left(n_{(a)}\cdot\tilde{v}\right) - \frac{1}{2} - 3(s_{(a)}\cdot n_{(a)})\left(s_{(a)}\cdot\tilde{v}\right) - \frac{2}{3}\left(n_{(a)}\cdot\tilde{v}\right)\right]\\ &\left.+\ \frac{\tilde{v}^k}{2r^4_{(a)}}\left[1 - 3(s_{(a)}\cdot n_{(a)})^2\right] + \frac{3s^k_{(a)}}{r^4_{(a)}}\left[\left(s_{(a)}\cdot\tilde{v}\right) - (s_{(a)}\cdot n_{(a)})\left(1 + 3n_{(a)}\cdot\tilde{v}\right)\right]\right\}\\ &+ O\left(\tilde{v}^2\right),\end{aligned} \tag{48}$$

and

$$\bar{h}_{(a),0} = \frac{n_{(a)}\cdot\tilde{v}}{r^2_{(a)}} + \frac{3J_2R^2_{(a)}}{r^4_{(a)}}\left[\frac{n_{(a)}\cdot\tilde{v}}{2} - \frac{5}{2}\left(n_{(a)}\cdot\tilde{v}\right)(s_{(a)}\cdot n_{(a)})^2\right] + O\left(\tilde{v}^2\right). \tag{49}$$

By taking into account the target accuracy of Gaia (see [6, 15]), the velocity contributions for an oblate body should be neglected. However, for sake of consistency and completeness with the assumptions adopted in this work, neglecting, *a priori*, terms which are part of the solution is not justified, even if the application to Gaia will surely dismiss many of them. Probably a Gaia-like mission that achieves a few sub-microarseconds in accuracy will benefit from these analytical contributions, especially in regards to a cross-checking comparison between different approaches. In this case it would be better to consider a metric which properly contemplates all the complexities of a non-spherical gravitational body; that, at the moment, is out of the scope of the present paper and deserves a dedicated work (see, e.g., [14]).

Therefore, the RAMOD reduced master equations which take into account the quadrupole structure for the *a*th single source consistently with the notation used above finally become:

- RAMOD3aQ (R3aQ)

$$\frac{d\bar{\ell}^k_{(a)}}{d\sigma} = (\mathrm{R3a})^k + \frac{2J_2R^2_{(a)}\mathcal{M}_{(a)}}{r^4_{(a)}}\left\{3\left[d^k_{n_{(a)}} - \frac{\bar{\ell}^k\left(\bar{\ell}\cdot n_a\right)}{2}\right]\left[\frac{5(s_{(a)}\cdot n_{(a)})^2}{2}\right.\right.$$
$$\left.+ 9(s_{(a)}\cdot n_{(a)})^2\left(n_{(a)}\cdot\tilde{v}\right) - \frac{1}{2} - 3(s_{(a)}\cdot n_{(a)})\left(s_{(a)}\cdot\tilde{v}\right) - \frac{2}{3}\left(n_{(a)}\cdot\tilde{v}\right)\right]$$
$$+ \frac{1}{2}\left[d^k_v - \frac{\bar{\ell}^k\left(\bar{\ell}\cdot\tilde{v}\right)}{2}\right]\left[1 - 3(s_{(a)}\cdot n_{(a)})^2\right] + 3\left[d^k_s - \frac{\bar{\ell}^k\left(\bar{\ell}\cdot s_a\right)}{2}\right]\left[\left(s_{(a)}\cdot\tilde{v}\right)\right.$$
$$\left.\left. - (s_{(a)}\cdot n_{(a)})\left(1 + 3\left(n_{(a)}\cdot\tilde{v}\right)\right)\right]\right\} + O\left(\tilde{v}^2\right) + O\left(h^2\right) \equiv (\mathrm{R3aQ})^k, \tag{50}$$

where we define $d^k_{s_{(a)}} = s^k_{(a)} - \bar{\ell}^k(\bar{\ell}\cdot s_{(a)})$.

In case of velocity equal to zero, the above equations become:

$$\frac{d\bar{\ell}^k_{(a)}}{d\sigma} - (\mathrm{R3s})^k = \frac{2J_2R^2_{(a)}\mathcal{M}_{(a)}}{r^4_{(a)}}\left\{\frac{3}{2}\left[d^k_{n_{(a)}} - \frac{\bar{\ell}^k\left(\bar{\ell}\cdot n_a\right)}{2}\right]\left[5(s_{(a)}\cdot n_{(a)})^2 - 1\right]\right.$$
$$\left. + 3\left[\frac{\bar{\ell}^k\left(\bar{\ell}\cdot s_a\right)}{2} - d^k_s\right](s_{(a)}\cdot n_{(a)})\right\} + O\left(h^2\right) \equiv (\mathrm{R3sQ})^k. \tag{51}$$

- RAMOD3bQ (R3bQ)

$$\frac{d\bar{\ell}^0_{(a)}}{d\sigma} = (\mathrm{R3b})^0 + \frac{3\mathcal{M}_{(a)}J_2R^2_{(a)}}{2r^4_{(a)}}\left[\frac{n_{(a)}\cdot\tilde{v}}{2} - \frac{5}{2}\left(n_{(a)}\cdot\tilde{v}\right)(s_{(a)}\cdot n_{(a)})^2\right]$$
$$+ O\left(\tilde{v}^2\right) + O\left(h^2\right) \equiv (\mathrm{R3bQ})^0, \tag{52}$$

$$\frac{d\bar{\ell}^k_{(a)}}{d\sigma} = (\mathrm{R3aQ})^k - \bar{\ell}^k(\mathrm{R3bQ})^0 + O\left(\tilde{v}^2\right) + O\left(h^2\right) \equiv (\mathrm{R3bQ})^k. \tag{53}$$

- RAMOD4aQ (R4aQ)

$$\frac{d\bar{\ell}^0_{(a)}}{d\sigma} = (\mathrm{R3bQ})^0 - \frac{4\mathcal{M}_{(a)}J_2\tilde{R}^2_{(a)}}{r^4_{(a)}}\left(\bar{\ell}\cdot\tilde{v}\right)\left\{\frac{3}{2}\left(\bar{\ell}\cdot n_a\right)\left[5(s_{(a)}\cdot n_{(a)})^2 - 1\right]\right.$$
$$\left. - 3\left(\bar{\ell}\cdot s_a\right)(s_{(a)}\cdot n_{(a)})\right\} + O\left(\tilde{v}^2\right) + O\left(h^2\right) \equiv (\mathrm{R4aQ})^0, \tag{54}$$

$$\frac{d\bar{\ell}^k_{(a)}}{d\sigma} = (\mathrm{R3bQ})^k - \frac{4\mathcal{M}_{(a)}J_2\tilde{R}^2_{(a)}}{r^4_{(a)}}\left\{\frac{3}{2}\left(\bar{\ell}\times n_a\times\tilde{v}\right)^k\left[5(s_{(a)}\cdot n_{(a)})^2 - 1\right]\right.$$
$$\left. + 3\left(\bar{\ell}\times\tilde{v}\times s_a\right)^k(s_{(a)}\cdot n_{(a)})\right\} + O\left(\tilde{v}^2\right) + O\left(h^2\right) \equiv (\mathrm{R4aQ})^k. \tag{55}$$

3. Light propagation through the Solar system and parametrized trajectories

When a photon approaches the weak gravitational field of the Solar System, heading to a Gaia-like observer, it will be subjected to the gravitational field generated by the mass of the bodies of the system while it will be rather insensitive to the contribution to the field due to their own motion. If one compares the scale of the Solar System and the photon crossing time through it—approximately 10 hours in total—the gravitational field of the Solar System cannot significantly change in a dynamical sense during such time, to the point that the source velocity can be considered constant all along the photon trajectory. This last remark facilitates the solution of the RAMOD equations.

Let us make explicit the vorticity of the congruence $\mathbf{u}$:

$$\begin{aligned}\omega_{\rho\sigma} &= P_\rho^\alpha(u)P_\sigma^\beta(u)\,\nabla_{[\alpha}u_{\beta]}\\ &= \nabla_{[\rho}u_{\sigma]} + u_{[\rho}\dot{u}_{\sigma]},\end{aligned} \tag{56}$$

where $P_\rho^\alpha(u) = \delta_\rho^\alpha + u^\alpha u_\rho$ is the metric induced on each hypersurface of simultaneity of $\mathbf{u}$. Considering that $u^\alpha u_\alpha = -1$, and $u^\alpha \nabla_\alpha u^\beta = \dot{u}^\beta$, we deduce:

$$\omega_{\rho\sigma} = -\eta_{0[\rho}\partial_{\sigma]}h_{00} + \partial_{[\rho}h_{\sigma]0} + \partial_0\Big(\eta_{0[\rho}h_{\sigma]0}\Big)$$

which implies

$$\begin{aligned}\omega_{00} &= 0,\\ \omega_{0i} &= 0,\\ \omega_{ij} &= \partial_{[i}h_{j]0}.\end{aligned} \tag{57}$$

Taking into account the metric (19), equations (57) show that if we want a vanishing vorticity we have to choose $h\vec{\nabla}\times\vec{v} + \vec{\nabla}(h)\times\vec{v} = 0$, which is satisfied if the velocity of the source is zero, i.e. a static case, or is constant, which corresponds to the case of our Solar System as mentioned above.

Now, within the scale of a vorticity-free geometry, from the Frobenius theorem (see [13]), the space-time can be foliated and one can always map the whole geodesic onto the hypersurface of simultaneity of the local barycentric observer at the time of observation. In this case the mapped trajectory can be expressed in a parametrized form with respect to the centre-of-mass (CM) of the gravitationally bounded system [11]:

$$x^i = \hat{\xi}^i + \int_o^{\hat{\tau}} \bar{\ell}^i \mathrm{d}\hat{\tau}, \tag{58}$$

where

- $\hat{\xi}^i$ is the impact parameter with respect to the CM of the gravitationally bounded system, i.e., $\delta_{ij}\bar{\ell}^i\hat{\xi}^j = 0 + \mathcal{O}(h)$ at the point of the closest approach with modulus $\hat{\xi} = \delta_{ij}\hat{\xi}^i\hat{\xi}^j$;
- $\hat{\tau} = \sigma - \hat{\sigma}$, being $\hat{\sigma}$ the value of geodesic parameter at the point of the closest approach.

Furthermore, if we approximate the quantity $\bar{\ell}^i$ in terms of small perturbations with respect to the unperturbed light direction $\bar{\ell}_\emptyset$:

$$\bar{\ell}^i = \bar{\ell}^i_\emptyset + \delta\bar{\ell}^i + \left(\delta\bar{\ell}^i\right)^2 + \ldots, \tag{59}$$

we note that the term $(\delta\bar{\ell}^i)$ has not to be integrated in the right-hand side of master equations (2) and (3) because it is of the order of the deflection, i.e. ϵ^2 or $O(h)$ (see also [16]). This implies that equation (58), for our purpose, can be approximated as:

$$x^i = \hat{\xi}^i + \left(\bar{\ell}^i_{\emptyset} + \delta\bar{\ell}^i\right)\hat{\tau} + O\left(h^2\right). \tag{60}$$

Following these assumptions, the approximated retarded distance can be parametrized as

$$r^i\left(\sigma, \tilde{\sigma}'\right) \approx \hat{\xi}^i + \left(\bar{\ell}^i_{\emptyset} + \delta\bar{\ell}^i\right)\hat{\tau} - \tilde{x}^i(\tilde{\sigma}) - \tilde{v}^i(\tilde{\sigma})r(\sigma, \tilde{\sigma}), \tag{61}$$

and the distance $r^i_{(a)}$ can be reformulated as

$$r^i\left(\sigma, \tilde{\sigma}'\right) \approx r^i\left(\hat{\tau}\right) - \tilde{v}(\tilde{\sigma})^i \hat{r}, \tag{62}$$

where

$$r^i\left(\hat{\tau}\right) = \hat{\xi}^i + \left(\bar{\ell}^i_{\emptyset} + \delta\bar{\ell}^k\right)\hat{\tau} - \tilde{x}^i(\tilde{\sigma}) = \hat{r}_p + \left(\bar{\ell}^i_{\emptyset} + \delta\bar{\ell}^k\right)\hat{\tau} \tag{63}$$

is the relative distance on the slice at the time of observation without the contribution of the source velocity, while

$$\hat{r}^i_p = \hat{\xi}^i - \tilde{x}^i \tag{64}$$

is the relative distance between the point of maximum approach of the photon to the CM and the source CM. Figure 1 sketches the relationship among these vectors on the slice corresponding to the time of observation.

By using the scalar $(\cdot)$ and vectorial $(\times)$ products and the parametrization, the impact parameter (35) with respect to the source becomes:

$$\mathrm{d}\left(\hat{\tau}\right)^k = \left[\bar{\ell} \times \left(r\left(\hat{\tau}\right) \times \bar{\ell}\right)\right]^k. \tag{65}$$

Note that the previous relation does not depend on $\hat{\tau}$ by definition, so it coincides with

$$d^k_p = \left[\bar{\ell} \times \left(\hat{r}_p \times \bar{\ell}\right)\right]^k \tag{66}$$

or, with the approximation (59),

$$d^k_p = \hat{r}^k_p - \bar{\ell}^k_{\emptyset}\left(\hat{r}_p \cdot \bar{\ell}_{\emptyset}\right), \tag{67}$$

with modulus $d_p = \sqrt{\mathbf{d}_p \cdot \mathbf{d}_p}$. Moreover, we can assume that $\hat{\tau}/d_p \approx 1/\tan\chi$, where χ is the angle between the direction at the observer towards the point of maximum approach to the source (along the unperturbed light direction) and its CM. In this case it is

$$r\left(\hat{\tau}\right) \approx \frac{d_p}{\sin\chi}\sqrt{1 + \frac{\left(\bar{\ell}_{\emptyset} \cdot \hat{r}_p\right)^2}{d_p^2}\sin\chi^2 + 2\frac{\left(\bar{\ell}_{\emptyset} \cdot \hat{r}_p\right)}{d_p}\cos\chi\,\sin\chi}\,. \tag{68}$$

4. Light deflection by spherical and oblate spheroidal gravitational sources with constant velocity

Condition (57) constrains the solution of the geodesic equation to specific circumstances. In particular we do not consider equations $(\mathrm{R4a})^k$ and $(\mathrm{R4aQ})^k$ since they derive from the terms $\partial_{[i}h_{j]0}$, which are null because of our physical assumption that the sources move with constant velocity. As far as the light deflection is concerned, we expect that the velocity contributions become relevant in affecting light propagation in the case of close approach when general relativistic effects become of the order of Gaia's expected accuracy together with the

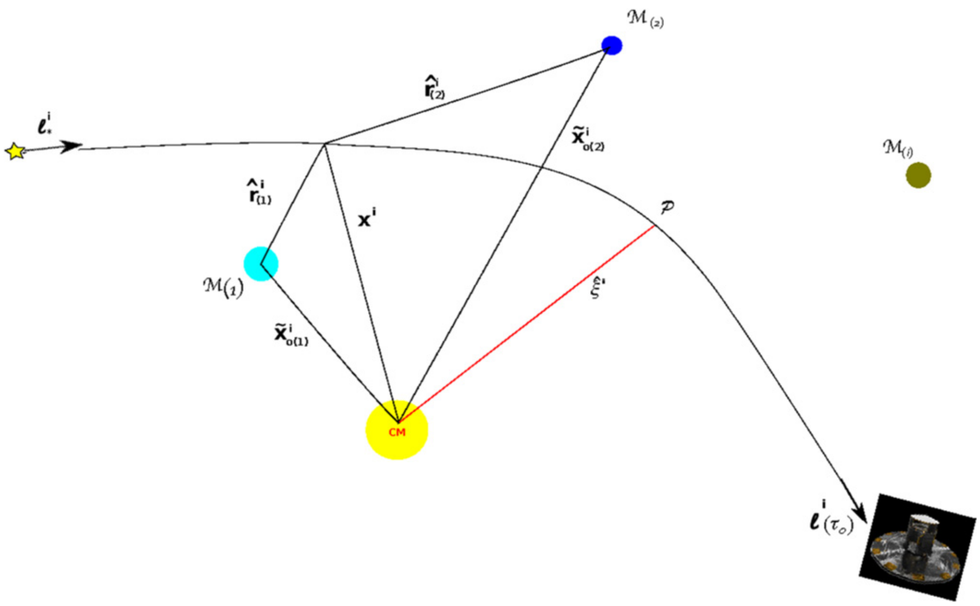

Figure 1. Mapped trajectories and relative source positions $\hat{r}^i_{(a)} = x^i - \tilde{x}^i_{(a)}$ on the hypersurface corresponding to the time of observation τ_o, where x^i and $\tilde{x}^i$ are the positions, respectively, of the photon and the gravitational source (colored discs $\mathcal{M}_{(1)}$, $\mathcal{M}_{(2)}$, etc) with respect to the centre-of-mass CM (yellow disc) of the gravitationally bounded system; $\hat{\xi}^i$ is the impact parameter of the light trajectory with respect to the CM and $\mathcal{P}$ the point of maximum approach of the photon to the CM; $\bar{\ell}^i_*$ is the light direction at the star and $\bar{\ell}^i(\tau_o)$ is the detected light direction at the observer.

multipolar structure of the source. In this section we proceed to make the solutions for light deflection explicit according to the RAMOD classification.

4.1. Monopole contribution without velocity: R3s case

This solution includes only the spatial derivative of the metric as in the static case.

Then equation (38) can be integrated as follows:

$$\Delta\bar{\ell}^k_{R3s} = 2\sum_a \mathcal{M}_{(a)}\left\{\int_{\hat{\tau}}^{\hat{\tau}_o}\left[\frac{1}{2}\bar{\ell}^k_{\emptyset}\left(\bar{\ell}_{\emptyset}\cdot r\right) - d^k_p\right]\frac{\mathrm{d}\hat{\tau}}{r^3}\right\} + O\left(h^2\right), \tag{69}$$

namely:

$$\Delta\bar{\ell}^k_{R3s} = 2\sum_a \mathcal{M}_{(a)}\left\{\left[\frac{1}{2}\bar{\ell}^k_{\emptyset}\left(\bar{\ell}_{\emptyset}\cdot\hat{r}_p\right) - d^k_p\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^3} + \frac{1}{2}\bar{\ell}^k_{\emptyset}\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{\hat{r}^3}\right\} + O\left(h^2\right), \tag{70}$$

i.e. (see table 1 for the list of the solved integrals)

$$\Delta\bar{\ell}^k_{R3s} = 2\sum_a \mathcal{M}_{(a)}\frac{1}{d_p^2}\left\{\left[\frac{1}{2}\bar{\ell_{\emptyset}}^k\left(\bar{\ell_{\emptyset}}\cdot\hat{r}_p\right) - d^k_p\right]\left[\left(\bar{\ell}_{\emptyset}\cdot n\right)\right]_{\hat{\tau}}^{\hat{\tau}_o} - \frac{1}{2}\bar{\ell_{\emptyset}}^k\left[\left(n\cdot\hat{r}_p\right)\right]_{\hat{\tau}}^{\hat{\tau}_o}\right\}, \tag{71}$$

Table 1. List of the recurrent integrals. They are named according to the order of magnitude in $r(\hat{\tau})$ in the numerator and denominator of the integral, indicated in the subscripts respectively on the left-side and on the right-side of the slash.

Notation	Corresponding expressions
$r(\hat{\tau})$	$\sqrt{\hat{r}_p^2 + 2(\bar{\ell}_{\emptyset} \cdot \hat{r}_p)\hat{\tau} + \hat{\tau}^2}$
$I_{0/0}$	$\arctan[(\bar{\ell}_{\emptyset} \cdot r)/d_p]$
$I_{1/0}$	$\mathrm{Log}(r \cdot \bar{\ell}_{\emptyset} + r)$
$I_{0/1}$	$1/r$
$I_{0/2}$	$1/r^2$
$I_{0/3}$	$1/r^3$
$I_{0/4}$	$1/r^4$
$I_{0/5}$	$1/r^5$
$I_{0/6}$	$1/r^6$
$I_{1/1}$	$(\bar{\ell}_{\emptyset} \cdot r)/r$
$I_{1/2}$	$(\bar{\ell}_{\emptyset} \cdot r)/r^2$
$I_{1/3}$	$(\bar{\ell}_{\emptyset} \cdot r)/r^3$
$I_{1/4}$	$(\bar{\ell}_{\emptyset} \cdot r)/r^4$
$I_{1/5}$	$(\bar{\ell}_{\emptyset} \cdot r)/r^5$
$I_{1/6}$	$(\bar{\ell}_{\emptyset} \cdot r)/r^6$
$I_{2/1}$	$(r \cdot \hat{r}_p)/r$
$I_{2/2}$	$(r \cdot \hat{r}_p)/r^2$
$I_{2/3}$	$(r \cdot \hat{r}_p)/r^3$
$I_{2/4}$	$(r \cdot \hat{r}_p)/r^4$
$I_{2/5}$	$(r \cdot \hat{r}_p)/r^5$
$I_{2/6}$	$(r \cdot \hat{r}_p)/r^6$

which reduces to

$$\Delta\bar{\ell}^k_{R3s} = 2\sum_a \mathcal{M}_{(a)} \left\{ -\frac{\bar{\ell}_{\emptyset}{}^k}{2}\left[\frac{1}{r}\right]_{\hat{\tau}}^{\hat{\tau}_o} - \frac{d_p^k}{d_p^2}\left[\bar{\ell}_{\emptyset} \cdot n\right]_{\hat{\tau}}^{\hat{\tau}_o} \right\} + O\left(h^2\right). \tag{72}$$

In the case of a source placed at infinity ($\bar{\ell}^k_{\emptyset} \equiv \sigma^k$):

$$\Delta\bar{\ell}^k_{R3s} \approx 2\sum_a \mathcal{M}\left\{ -\frac{\sigma^k}{2r\left(\hat{\tau}_o\right)} - \frac{d_p^k}{d_p^2}\left[1 + \sigma \cdot n\left(\hat{\tau}_o\right)\right]\right\} + O\left(h^2\right), \tag{73}$$

if the observable shift with respect to the initial direction at infinity is $\delta\sigma^k = c^{-1}[\sigma \times \Delta\dot{x} \times \sigma]^k$, we get

$$\delta\sigma^k \approx 2\sum_a \mathcal{M}\left\{ -\frac{d_p^k}{d_p^2}\left[1 + \sigma \cdot n\left(\hat{\tau}_o\right)\right]\right\} + O\left(h^2\right). \tag{74}$$

Formula (74) recovers the one used by Klioner (2003).

4.1.1. Monopole light deflection due to one body. Let us apply equation (38) to the computation of the light deflection due to one body, for example, the Sun, and check if we recover the results in literature for grazing rays [17].

With respect to the local barycentric observer $u^\alpha = e^\phi \delta_0^\alpha$, the total light deflection is given by the modulus:

$$\Delta\bar{\ell} = \sqrt{P(u)_{\alpha\beta}\Delta\bar{\ell}^\alpha\Delta\bar{\ell}^\beta} = \sqrt{\left(g_{\alpha\beta} + u_\alpha u_\beta\right)\Delta\bar{\ell}^\alpha\Delta\bar{\ell}^\beta}. \tag{75}$$

We have to consider only the Euclidean metric δ_{ij}, since $\Delta\bar{\ell}^k$ is of the order of h. Moreover the definition (75) implies a projection with respect to u^α, and in the static case $\bar{\ell}^0 = 0$; therefore, we limit the integration to the following expression

$$\Delta\bar{\ell}^k = 2\mathcal{M}^{(\mathrm{Sun})}\left[\frac{1}{2}\int_{-\infty}^{\infty}\frac{\bar{\ell}_o^k\hat{\tau}}{\left(\hat{\xi}^2 + \hat{\tau}^2\right)^{3/2}}\mathrm{d}\hat{\tau} - \int_{-\infty}^{\infty}\frac{\hat{\xi}^k}{\left(\hat{\xi}^2 + \hat{\tau}^2\right)^{3/2}}\mathrm{d}\hat{\tau}\right], \tag{76}$$

which yields the well known solution:

$$\Delta\bar{\ell} = \frac{4}{\hat{\xi}}\mathcal{M}^{(\mathrm{Sun})}. \tag{77}$$

We can also check the validity of our assumption by limiting the integration at the observer. Considering the source at infinity, from (68)

$$\begin{aligned}\Delta\bar{\ell}_{R3s}^k = 2\sum_a \mathcal{M}_{(a)}\Bigg\{&-\frac{\bar{\ell}_\emptyset^k}{2}\left[\frac{\sin\chi}{d_p\sqrt{1 + \frac{\left(\bar{\ell}_\emptyset\cdot\hat{r}_p\right)^2}{d_p^2}\sin\chi^2 + 2\frac{\left(\bar{\ell}_\emptyset\cdot\hat{r}_p\right)}{d_p}\cos\chi\,\sin\chi}}\right] \\ &- \frac{d_p^k}{d_p^2}\left[\frac{\left(\bar{\ell}_\emptyset\cdot\hat{r}_p + \hat{\tau}_o\right)\sin\chi}{d_p\sqrt{1 + \frac{\left(\bar{\ell}_\emptyset\cdot\hat{r}_p\right)^2}{d_p^2}\sin\chi^2 + 2\frac{\left(\bar{\ell}_\emptyset\cdot\hat{r}_p\right)}{d_p}\cos\chi\,\sin\chi}} + 1\right]\Bigg\} + O\left(h^2\right),\end{aligned} \tag{78}$$

and in the limit $\chi \ll 1$, since $\hat{\tau}_o/d_p = \cos\chi/\sin\chi$, we obtain

$$\Delta\bar{\ell}_{R3s}^k \approx 2\sum_a \mathcal{M}_{(a)}\left\{-2\frac{d_p^k}{d_p^2}\right\} + O\left(h^2\right). \tag{79}$$

Then the modulus in the case of one body is again:

$$\Delta\bar{\ell} = \frac{4}{\hat{\xi}}\mathcal{M}. \tag{80}$$

These results check the validity of RAMOD master equation when it is applied to one single body and make us confident to proceed with the solution for a n-body system.

4.2. Monopole and velocity contribution: R3a case

This solution includes the perturbations depending on the spatial derivative of the h term and the retarded distance approximation. According to the assumption of section 3 we consider also the velocity of the source constant, i.e. not dependent on $\hat{\tau}$. For sake of simplicity, let us denote:

$$D_p^k = d_p^k - \frac{\bar{\ell}_\emptyset^k \left(\bar{\ell}_\emptyset \cdot \hat{r}_p\right)}{2}, \tag{81}$$

$$D_v^k = d_v^k - \frac{\bar{\ell}_\emptyset^k \left(\bar{\ell}_\emptyset \cdot v\right)}{2}. \tag{82}$$

In the case discussed here we limit ourselves to the integration of the following terms:

$$\Delta\bar{\ell}_{R3a}^k \approx \Delta\bar{\ell}_{R3s}^k + 2\sum_a \mathcal{M}_{(a)}\left\{-2D_p^k\left(\hat{r}_p \cdot \tilde{v}\right)\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^4} + \left[-2D_p^k\left(\bar{\ell}_\emptyset \cdot \tilde{v}\right) + \bar{\ell}_\emptyset^k\left(\hat{r}_p \cdot \tilde{v}\right)\right]\right.$$
$$\left.\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^4} + \bar{\ell}_\emptyset^k\left(\bar{\ell}_\emptyset \cdot \tilde{v}\right)\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^2\mathrm{d}\hat{\tau}}{r^4} + D_v^k\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^2}\right\}. \tag{83}$$

In addition to the static case solution whose form appears in equation (72), the solution of the R3a master equation is:

$$\Delta\bar{\ell}_{R3a}^k = \Delta\bar{\ell}_{R3s}^k + 2\sum_a \frac{\mathcal{M}_{(a)}}{d_p^2}\left\{ -d_p^2\frac{\bar{\ell}_\emptyset^k}{2}\left(\bar{\ell}_\emptyset \cdot \tilde{v}\right)\left[\frac{\hat{\tau}}{r^2}\right]_{\hat{\tau}}^{\hat{\tau}_o} - D_p^k\left(\hat{r}_p \cdot \tilde{v}\right)\left[\frac{\left(\bar{\ell}_\emptyset \cdot n\right)}{r}\right]_{\hat{\tau}}^{\hat{\tau}_o}\right.$$
$$+\left[D_v^k d_p - \frac{D_p^k}{d_p}\left(d_p \cdot \tilde{v}\right) + \frac{\bar{\ell}_\emptyset^k}{2d_p}\left(\left(\tilde{v} \times \hat{r}_p\right)\cdot\left(\hat{r}_p \times \bar{\ell}_\emptyset\right)\right)\right]\left[\arctan\left(\frac{\bar{\ell}_\emptyset \cdot r}{d_p}\right)\right]_{\hat{\tau}}^{\hat{\tau}_o}$$
$$\left.+\left[D_p^k\left(\bar{\ell}_\emptyset \cdot \tilde{v}\right) - \frac{\bar{\ell}_\emptyset^k}{2}\left(d_p \cdot \tilde{v}\right)\right]\left[\frac{\left(\hat{r}_p \cdot n\right)}{r}\right]_{\hat{\tau}}^{\hat{\tau}_o}\right\} + O(h) + O\left(v^2\right). \tag{84}$$

If the source is at infinity and in the limit of $\chi \ll 1$, i.e. at the observer and for grazing rays, these contributions vanish in the case of one body.

4.3. Monopole and velocity contribution: R3b case

This solution includes the perturbations depending on the time derivative of the h term according to the retarded distance approximation. Assuming the source velocity is constant, we have to integrate:

$$\Delta\bar{\ell}_{R3b}^0 \approx \sum_a \mathcal{M}_{(a)}\left\{\left(\hat{r}_p \cdot \tilde{v}\right)\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^3} + \left(\bar{\ell}_\emptyset \cdot \tilde{v}\right)\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^3}\right\}, \tag{85}$$

$$\Delta\bar{\ell}_{R3b}^k \approx \Delta\bar{\ell}_{R3a}^k - \bar{\ell}_\emptyset^k\Delta\bar{\ell}_{R3b}^0. \tag{86}$$

The solution is straightforwardly:

$$\Delta\bar{\ell}_{R3b}^0 \approx \sum_a \frac{\mathcal{M}_{(a)}}{d_p^2}\left\{\left(\hat{r}_p \cdot \tilde{v}\right)\left[\left(\bar{\ell}_\emptyset \cdot n\right)\right]_{\hat{\tau}}^{\hat{\tau}_o} - \left(\bar{\ell}_\emptyset \cdot \tilde{v}\right)\left[\left(n \cdot \hat{r}_p\right)\right]_{\hat{\tau}}^{\hat{\tau}_o}\right\}, \tag{87}$$

$$\Delta\bar{\ell}_{R3b}^k \approx \Delta\bar{\ell}_{R3a}^k - \bar{\ell}_\emptyset^k\Delta\bar{\ell}_{R3b}^0. \tag{88}$$

For a stellar source at infinity and in the case of light rays grazing only one gravitational source, from table 1 these contributions result:

$$\Delta\bar{\ell}^{0}_{R3b} \approx 2\sum_{a}\mathcal{M}_{(a)}\frac{\left(\hat{r}_p\cdot\tilde{v}\right)}{d_p^2}, \tag{89}$$

$$\Delta\bar{\ell}^{k}_{R3b} \approx 4\sum_{a}\mathcal{M}_{(a)}\left\{\frac{d_p^k}{d_p^2}\left[1+\frac{\left(\bar{\ell}_{\emptyset}\cdot\hat{n}_p\right)}{2}\right]-\bar{\ell}^k_{\emptyset}\frac{\left(\hat{r}_p\cdot\tilde{v}\right)}{2d_p^2}\right\}. \tag{90}$$

4.4. Monopole and velocity contribution: R4a case

The (R4a)^k equations include the perturbations depending on the spatial derivative of the h_{0i} term, i.e. those depending on the mass-current contribution. According to the constraint on the vorticity imposed by the physical assumption as discussed in section (3), we have to consider only equation (R4a)^0:

$$\Delta\bar{\ell}^{0}_{R4a} \approx \Delta\bar{\ell}^{0}_{R3b} + 4\sum_{a}\mathcal{M}_{(a)}\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\left[\left(\bar{\ell}_{\emptyset}\cdot\hat{r}_p\right)\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^3}+\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{\hat{r}^3}\right], \tag{91}$$

$$\Delta\bar{\ell}^{k}_{R4a} \approx \Delta\bar{\ell}^{k}_{R3b}, \tag{92}$$

i.e.

$$\Delta\bar{\ell}^{0}_{R4a} \approx \Delta\bar{\ell}^{0}_{R3b} + 4\sum_{a}\frac{\mathcal{M}_{(a)}}{d_p^2}\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\left\{\left(\bar{\ell}_{\emptyset}\cdot\hat{r}_p\right)\left[\left(\bar{\ell}_{\emptyset}\cdot n\right)\right]_{\hat{\tau}}^{\hat{\tau}_o}-\left[\left(n\cdot\hat{r}_p\right)\right]_{\hat{\tau}}^{\hat{\tau}_o}\right\}, \tag{93}$$

$$\Delta\bar{\ell}^{k}_{R4a} \approx \Delta\bar{\ell}^{k}_{R3b}. \tag{94}$$

Again, as for R3b case, considering a source placed at infinity and in the limit of a grazing light ray, we get at the observer:

$$\Delta\bar{\ell}^{0}_{R4a} \approx 2\sum_{a}\frac{\mathcal{M}_{(a)}}{d_p^2}\left\{\left(\hat{r}_p\cdot\tilde{v}\right)+4\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\left(\bar{\ell}_{\emptyset}\cdot\hat{r}_p\right)\right\}, \tag{95}$$

$$\Delta\bar{\ell}^{k}_{R4a} \approx 4\sum_{a}\frac{\mathcal{M}_{(a)}}{d_p^2}\left\{d_p^k\left[1+\frac{\left(\bar{\ell}_{\emptyset}\cdot\hat{n}_p\right)}{2}\right]-\bar{\ell}^k_{\emptyset}\frac{\left(\hat{r}_p\cdot\tilde{v}\right)}{2}\right\}. \tag{96}$$

4.5. Light deflection by a static oblate body: R3sQ case

For sake of convenience, from now on we drop the suffix (a). Considering that the quadrupole structure of the source may be relevant in affecting light propagation only during close approach, here and after, to compute the quadrupole contribution, we refer only to a single body. Moreover, as in section 5.2, we denote:

$$D_s^k = d_s^k - \frac{\bar{\ell}^k_{\emptyset}\left(\bar{\ell}_{\emptyset}\cdot s\right)}{2}, \tag{97}$$

where s^k is the axis of the source normal to its equatorial plane. Equation R3sQ (51) can be integrated with respect to the parametrized trajectory as follows

$$\begin{aligned}\Delta\bar{\ell}^k_{R3sQ} = 3J_2R^2\mathcal{M}\Bigg\{ & -\Big[D^k_p + 2\big(s\cdot\hat{r}_p\big)D^k_s\Big]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^5} + 5\big(s\cdot\hat{r}_p\big)^2 D^k_p\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{\hat{r}^7}\\ & + \Big[\bar{\ell}^k_\text{ø} - 2\big(s\cdot\bar{\ell}_\text{ø}\big)D^k_s\Big]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^5} + 5\big(s\cdot\hat{r}_p\big)\Big[2\big(s\cdot\bar{\ell}_\text{ø}\big)D^k_p\\ & - \bar{\ell}^k_\text{ø}\big(s\cdot\hat{r}_p\big)\Big]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^7} + 5\big(s\cdot\bar{\ell}_\text{ø}\big)\Big[\big(s\cdot\bar{\ell}_\text{ø}\big)D^k_p - \bar{\ell}^k_\text{ø}\big(s\cdot\hat{r}_p\big)\Big]\\ & \times\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^2\mathrm{d}\hat{\tau}}{r^7} - 5\bar{\ell}^k_\text{ø}\big(s\cdot\bar{\ell}_\text{ø}\big)^2\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^3\mathrm{d}\hat{\tau}}{r^7}\Bigg\} + O\big(h^2\big), \end{aligned} \tag{98}$$

namely (for the details of the solution see table 1 and the appendix),

$$\begin{aligned}\Delta\bar{\ell}^k_{R3sQ} = 6J_2R^2\frac{\mathcal{M}}{d_p^2}\Big\{ & \big[I_{0/3}\big]_{\hat{\tau}}^{\hat{\tau}_o}C^k_1 + \big[\hat{\tau}I_{0/3}\big]_{\hat{\tau}}^{\hat{\tau}_o}C^k_2 + \big[\hat{\tau}I_{0/5}\big]_{\hat{\tau}}^{\hat{\tau}_o}C^k_3 + \big[I_{1/1}\big]_{\hat{\tau}}^{\hat{\tau}_o}C^k_4\\ & + \big[I_{1/3}\big]_{\hat{\tau}}^{\hat{\tau}_o}C^k_5 + \big[I_{1/5}\big]_{\hat{\tau}}^{\hat{\tau}_o}C^k_6 + \big[I_{2/3}\big]_{\hat{\tau}}^{\hat{\tau}_o} + C^k_7\big[I_{2/5}\big]_{\hat{\tau}}^{\hat{\tau}_o}C^k_8\Big\} + O\big(h^2\big). \end{aligned} \tag{99}$$

If we consider a grazing light ray emitted by a star located at infinity, the above equations reduce to:

$$\Delta\bar{\ell}^k_{R3sQ} = 6J_2R^2\frac{\mathcal{M}}{d_p^2}\big\{2C^k_4\big\} + O\big(h^2\big). \tag{100}$$

Similarly to the spherical case, let us compute the total light deflection nearby a single gravity source. In this case, we expect that our formula reduces to the available expressions known in the literature [14, 15, 18, 19]. For this scope we can assume:

- $d^k_p = \hat{r}^k_p \equiv \xi^k$ the radial vector;
- $d^k_p/d_p = -n^k$ the unit radial vector;
- $\bar{\ell}^k_\text{ø} \equiv t^k$ the tangential vector to the line-of-sight;
- $\sqrt{\xi^i\xi_i} \equiv d_p$ the impact parameter;
- and $m^k = (\bar{\ell}\times n)^k$ the orthoradial direction.

Then:

$$\Delta\bar{\ell}^k_{R3sQ} = 4J_2R^2\frac{\mathcal{M}}{d_p^3}\Big\{n^k\Big[1-(s\cdot t)^2 - 4(s\cdot n)^2\Big] + 2(s\cdot n)\Big[s^k - t^k(s\cdot t)\Big]\Big\} + O\big(h^2\big), \tag{101}$$

and, by expressing the vector s^k as linear combination in terms of the orthonormal basis $(\mathbf{t}, \mathbf{n}, \mathbf{m})$

$$s^k = (s\cdot t)t^k + (s\cdot n)n^k + (s\cdot m)m^k, \tag{102}$$

we obtain the same formula deduced in Crosta and Mignard [19]:

$$\begin{aligned}\Delta\bar{\ell}^k &= \Delta\bar{\ell}^k_{R3s} + \Delta\bar{\ell}^k_{R3sQ}\\ &= \frac{4\mathcal{M}}{d_p}\left\{\left[1 + \frac{J_2R^2}{d_p^2}\Big(1-(s\cdot t)^2 - 2(s\cdot n)^2\Big)\right]n^k + \frac{J_2R^2}{d_p^2}(s\cdot m)(s\cdot n)m^k\right\}. \end{aligned} \tag{103}$$

4.6. Quadrupole contribution with velocity: R3aQ case

Equation (50) in function of the parametrized trajectory becomes:

$$
\begin{aligned}
&\Delta\bar{\ell}^k_{R3aQ} = \Delta\bar{\ell}^k_{R3a} + \Delta\bar{\ell}^k_{R3sQ}\\
&+ 2J_2R^2\mathcal{M}\left\{\left[\frac{1}{2}D^k_v + 3D^k_s(s\cdot\tilde{v})\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^4} - \left[D^k_p\left(9\left(s\cdot\hat{r}_p\right)(s\cdot\tilde{v})\right.\right.\right.\\
&\left.\left. + \ 2\left(\hat{r}_p\cdot\tilde{v}\right)\right) - \frac{3}{2}D^k_v\left(s\cdot\hat{r}_p\right)^2 + 3D^k_s\left(s\cdot\hat{r}_p\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^6}\\
&+ 27D^k_p\left(s\cdot\hat{r}_p\right)^2\left(\hat{r}_p\cdot\tilde{v}\right)\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^8} - \left[D^k_p\left(9\left(s\cdot\bar{\ell}_\emptyset\right)(s\cdot\tilde{v})\right.\right.\\
&\left. + \ 2\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right) - \bar{\ell}^k_\emptyset\left(9\left(s\cdot\hat{r}_p\right)(s\cdot\tilde{v}) + 2\left(\hat{r}_p\cdot\tilde{v}\right)\right)\\
&\left. + \ 3D^k_v\left(s\cdot\bar{\ell}_\emptyset\right)\left(s\cdot\hat{r}_p\right) + 3D^k_s\left(\left(\hat{r}_p\cdot\tilde{v}\right)\left(s\cdot\bar{\ell}_\emptyset\right) + \left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^6}\\
&+ 27\left(s\cdot\hat{r}_p\right)\left[D^k_p\left(\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right) + 2\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right) - \bar{\ell}^k_\emptyset\left(s\cdot\hat{r}_p\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^8}\\
&+ \left[9\bar{\ell}^k_\emptyset\left(s\cdot\bar{\ell}_\emptyset\right)(s\cdot\tilde{v}) + 2\bar{\ell}^k_\emptyset\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right) - 3D^k_s\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\left(\left(s\cdot\bar{\ell}_\emptyset\right) + \left(s\cdot\hat{r}_p\right)\right)\right.\\
&\left. + \ \frac{1}{2}D^k_v\left(s\cdot\bar{\ell}_\emptyset\right)^2\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^2\mathrm{d}\hat{\tau}}{r^6} + 27\left[\left(s\cdot\bar{\ell}_\emptyset\right)D^k_p\right.\\
&+ \left(2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right) + \left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right)\\
&- \bar{\ell}^k_\emptyset\left(s\cdot\hat{r}_p\right) + \left(\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right.\\
&\left.\left. + \ 2\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^2\mathrm{d}\hat{\tau}}{r^8} + 27\left(s\cdot\bar{\ell}_\emptyset\right)\left[D^k_p\left(s\cdot\bar{\ell}_\emptyset\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right.\\
&\left. - \ \bar{\ell}^k_\emptyset\left(2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right) + \left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^3\mathrm{d}\hat{\tau}}{r^8}\\
&\left. + \ 27\left[\bar{\ell}^k_\emptyset\left(s\cdot\bar{\ell}_\emptyset\right)^2\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^4\mathrm{d}\hat{\tau}}{r^8}\right\} + O\left(\tilde{v}^2\right) + O\left(h^2\right).
\end{aligned}
\tag{104}
$$

Namely, from table 1 and the appendix, we denote

$$
\begin{aligned}
\Delta\bar{\ell}^k_{R3aQ} = \Delta\bar{\ell}^k_{R3a} + \Delta\bar{\ell}^k_{R3sQ} + J_2R^2\frac{\mathcal{M}}{d_p^2}\Big\{&\left[I_{0/0}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_9 + \left[I_{0/4}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{10} + \left[\hat{\tau}I_{0/4}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{11}\\
&+ \left[I_{0/6}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{12} + \left[\tau I_{0/6}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{13} + \left[I_{1/2}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{14} + \left[I_{1/4}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{15} + \left[I_{1/6}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{16}\\
&+ \left[I_{2/4}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{17} + \left[I_{2/6}\right]_{\hat{\tau}}^{\hat{\tau}_o}C^k_{18}\Big\} + O\left(h^2\right) + O\left(\tilde{v}^2\right).
\end{aligned}
\tag{105}
$$

For a source at infinity and in the limit of $\chi \ll 1$ the above contributions vanish as in the case of a mass monopole moving with constant velocities.

4.7. Quadrupole contribution with velocity: R3bQ case

From equations (53) we have to solve

$$\Delta\bar{\ell}^0_{R3bQ} = \Delta\bar{\ell}^0_{R3b} + 3\mathcal{M}J_2R^2\Bigg\{\frac{\hat{r}_p\cdot\tilde{v}}{2}\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^5} + \frac{\bar{\ell}_\emptyset\cdot v}{2}\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^5}$$
$$- \frac{5}{2}\Big[\big(\hat{r}_p\cdot\tilde{v}\big)\big(s\cdot\hat{r}_p\big)^2\Big]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^7} - 5\big(s\cdot\hat{r}_p\big)\Big[\big(\hat{r}_p\cdot\tilde{v}\big)\big(s\cdot\bar{\ell}_\emptyset\big)$$
$$+ \frac{1}{2}\big(\bar{\ell}_\emptyset\cdot\tilde{v}\big)\big(s\cdot\hat{r}_p\big)\Big]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^7}$$
$$- 5\big(s\cdot\bar{\ell}_\emptyset\big)\Big[\frac{1}{2}\big(s\cdot\bar{\ell}_\emptyset\big)\big(\hat{r}_p\cdot\tilde{v}\big) + \big(\bar{\ell}_\emptyset\cdot\tilde{v}\big)\big(s\cdot\hat{r}_p\big)\Big]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^2\mathrm{d}\hat{\tau}}{r^7}$$
$$- \frac{5}{2}\Big[\big(s\cdot\bar{\ell}_\emptyset\big)^2\big(\bar{\ell}_\emptyset\cdot\tilde{v}\big)\Big]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^3\mathrm{d}\hat{\tau}}{r^7}\Bigg\} + O\big(\tilde{v}^2\big) + O\big(h^2\big), \tag{106}$$

$$\Delta\bar{\ell}^k_{R3bQ} = \Delta\bar{\ell}^k_{R3aQ} - \bar{\ell}^k_\emptyset(\mathrm{R3bQ})^0 + O\big(\tilde{v}^2\big) + O\big(h^2\big). \tag{107}$$

Expressing the above formulae in terms of the solutions of the integrals listed in table 1, these quantities read

$$\Delta\bar{\ell}^0_{R3bQ} = \Delta\bar{\ell}^0_{R3b} + \frac{\mathcal{M}J_2R^2}{d_p^2}\Big\{\big[I_{0/3}\big]_{\hat{\tau}}^{\hat{\tau}_o}C_{19} + \big[\hat{\tau}I_{0/3}\big]_{\hat{\tau}}^{\hat{\tau}_o}C_{20} + \big[\hat{\tau}I_{0/5}\big]_{\hat{\tau}}^{\hat{\tau}_o}C_{21} + \big[I_{1/1}\big]_{\hat{\tau}}^{\hat{\tau}_o}$$
$$C_{22} + \big[I_{1/3}\big]_{\hat{\tau}}^{\hat{\tau}_o}C_{23} + \big[I_{1/5}\big]_{\hat{\tau}}^{\hat{\tau}_o}C_{24} + \big[I_{2/3}\big]_{\hat{\tau}}^{\hat{\tau}_o}C_{25} + \big[I_{2/5}\big]_{\hat{\tau}}^{\hat{\tau}_o}C_{26}\Big\} + O\big(\tilde{v}^2\big)$$
$$+ O\big(h^2\big), \tag{108}$$

$$\Delta\bar{\ell}^k_{R3bQ} = \Delta\bar{\ell}^k_{R3aQ} - \bar{\ell}^k_\emptyset\Delta\bar{\ell}^0_{R3bQ} + O\big(\tilde{v}^2\big) + O\big(h^2\big). \tag{109}$$

For a stellar source at infinity and in the limit of light ray grazing the gravity source, we get:

$$\Delta\bar{\ell}^0_{R3bQ} = \Delta\bar{\ell}^0_{R3b} + \frac{2\mathcal{M}J_2R^2}{d_p^2}C_{22} + O\big(\tilde{v}^2\big) + O\big(h^2\big), \tag{110}$$

$$\Delta\bar{\ell}^k_{R3bQ} = \Delta\bar{\ell}^k_{R3aQ} - \bar{\ell}^k_\emptyset\Delta\bar{\ell}^0_{R3bQ} + O\big(\tilde{v}^2\big) + O\big(h^2\big). \tag{111}$$

With the same notations of the single static source, the above equations collapse to

$$\Delta\bar{\ell}^0_{R3bQ} = \Delta\bar{\ell}^0_{R3b} + \frac{\mathcal{M}J_2R^2}{d_p^3}\Big\{-(n\cdot\tilde{v})\Big[1 - 4(s\cdot n)^2 - (s\cdot t)^2\Big] + 2(s\cdot t)(s\cdot n)(t\cdot v)\Big\}$$
$$+ O\big(\tilde{v}^2\big) + O\big(h^2\big), \tag{112}$$

$$\Delta\bar{\ell}^k_{R3bQ} = \Delta\bar{\ell}^k_{R3aQ} + \frac{\mathcal{M}J_2R^2}{d_p^3}\Big\{t^k(n\cdot\tilde{v})\Big[1-4(s\cdot n)^2-(s\cdot t)^2\Big]$$
$$- 2t^k(s\cdot t)(s\cdot n)(t\cdot v)\Big\} + O\big(\tilde{v}^2\big) + O\big(h^2\big). \tag{113}$$

4.8. Quadrupole contribution with velocity: R4aQ solution

The integral expressions of equations (54) and (55) with the physical assumption of a source moving with constant velocity are:

$$\Delta\bar{\ell}^{0}_{R4aQ} = \Delta\bar{\ell}^{0}_{R3bQ} - 4\mathcal{M}_{(a)}J_2R^2_{(a)}\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\left\{-\left[\frac{1}{2}\left(\bar{\ell}_{\o}\cdot\hat{r}_p\right)+3\left(s\cdot\bar{\ell}_{\o}\right)\left(s\cdot\hat{r}_p\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^5}\right.$$
$$-\left[\frac{1}{2}+3\left(s\cdot\bar{\ell}_{\o}\right)^2\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^5}+\left[\frac{5}{2}\left(s\cdot\hat{r}_p\right)^2\left(\bar{\ell}_{\o}\cdot\hat{r}_p\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\mathrm{d}\hat{\tau}}{r^7}$$
$$+\frac{5}{2}\left(s\cdot\hat{r}_p\right)\left[2\left(s\cdot\bar{\ell}_{\o}\right)\left(\bar{\ell}_{\o}\cdot\hat{r}_p\right)+\left(s\cdot\hat{r}_p\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}\mathrm{d}\hat{\tau}}{r^7}$$
$$+\frac{5}{2}\left(s\cdot\bar{\ell}_{\o}\right)\left[\left(\bar{\ell}_{\o}\cdot\hat{r}_p\right)\left(s\cdot\bar{\ell}_{\o}\right)+2\left(s\cdot\hat{r}_p\right)\right]\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^2\mathrm{d}\hat{\tau}}{r^7}$$
$$\left.+\frac{5}{2}\left(s\cdot\bar{\ell}_{\o}\right)^2\int_{\hat{\tau}}^{\hat{\tau}_o}\frac{\hat{\tau}^3\mathrm{d}\hat{\tau}}{r^7}\right\}+O\left(\tilde{v}^2\right)+O\left(h^2\right), \tag{114}$$

$$\Delta\bar{\ell}^{k}_{R3bQ} = \Delta\bar{\ell}^{k}_{R4aQ} + O\left(\tilde{v}^2\right)+O\left(h^2\right). \tag{115}$$

Considering table 1 and the appendix, the solution reads:

$$\Delta\bar{\ell}^{0}_{R4aQ} = \Delta\bar{\ell}^{0}_{R3bQ} - \frac{2\mathcal{M}_{(a)}J_2R^2_{(a)}}{d_p^2}\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\left\{\left[I_{0/3}\right]_{\hat{\tau}}^{\hat{\tau}_o}C_{27}+\left[\hat{\tau}I_{0/3}\right]_{\hat{\tau}}^{\hat{\tau}_o}C_{28}+\left[\hat{\tau}I_{0/5}\right]_{\hat{\tau}}^{\hat{\tau}_o}\right.$$
$$\left.C_{29}+\left[I_{1/1}\right]_{\hat{\tau}}^{\hat{\tau}_o}C_{30}+\left[I_{1/3}\right]_{\hat{\tau}}^{\hat{\tau}_o}C_{31}+\left[I_{1/5}\right]_{\hat{\tau}}^{\hat{\tau}_o}C_{32}+\left[I_{2/3}\right]_{\hat{\tau}}^{\hat{\tau}_o}C_{33}+\left[I_{2/5}\right]_{\hat{\tau}}^{\hat{\tau}_o}C_{34}\right\}$$
$$+O\left(\tilde{v}^2\right)+O\left(h^2\right), \tag{116}$$

$$\Delta\bar{\ell}^{k}_{R4aQ} = \Delta\bar{\ell}^{k}_{R3bQ} + O\left(\tilde{v}^2\right)+O\left(h^2\right). \tag{117}$$

For a source located at infinity and in the limit of the small angle χ between the directions at the observer towards the body CM and the point of closest approach of the light trajectory, we obtain:

$$\Delta\bar{\ell}^{0}_{R4aQ} = \Delta\bar{\ell}^{0}_{R3bQ} - \frac{4\mathcal{M}_{(a)}J_2R^2_{(a)}}{d_p^2}\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)C_{30}+O\left(\tilde{v}^2\right)+O\left(h^2\right), \tag{118}$$

$$\Delta\bar{\ell}^{k}_{R4aQ} = \Delta\bar{\ell}^{k}_{R4aQ} + O\left(\tilde{v}^2\right)+O\left(h^2\right). \tag{119}$$

Again, with the same conventions of the static source, for the time component we deduce finally for the light deflection:

$$\Delta\bar{\ell}^{0}_{R4aQ} = \Delta\bar{\ell}^{0}_{R3bQ} + \frac{16\mathcal{M}_{(a)}J_2R^2_{(a)}}{d_p^3}(t\cdot\tilde{v})(s\cdot n)(s\cdot t)+O\left(\tilde{v}^2\right)+O\left(h^2\right). \tag{120}$$

5. Photon trajectory

The first integration of RAMOD equations gives the estimates of the deflection effects and the left-hand side results $\mathrm{d}x^\alpha/\mathrm{d}\hat{\tau} - \bar{\ell}^\alpha_o$. The contributions to $\bar{\ell}^\alpha$ obtained with this first integration gives the quantity needed to solve the astrometric problem, namely the sphere reconstruction by the cosine of the observed angle between the light direction and the spatial axes of observer's tetrad. A different kind of work can be done with the second integration which solves the ray tracing for the photon emitted by a star and intercepted at the observer's location, thus obtaining the quantity $\Delta x^\alpha = x^\alpha(\hat{\tau}_o) - x^\alpha(\hat{\tau}_*)$ at the different approximation levels. This quantity is needed to address the problem of a relativistic consistent astrometric parameters that can be used for further applications in the on-going work needed for the Gaia sphere reconstruction.

5.1. Monopoles

In the case of monopoles with constant velocities, the integral to be solved is the following:

$$\Delta x^k_{R3s} = \bar{\ell}^k_o \Delta\hat{\tau} + 2\sum_a \mathcal{M}_{(a)} \left\{ -\frac{\bar{\ell}_\emptyset^{\,k}}{2} \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{0/1} \mathrm{d}\hat{\tau} - \frac{d_p^k}{d_p^2} \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{1/1} \mathrm{d}\hat{\tau} \right\} + O\left(h^2\right), \tag{121}$$

$$\begin{aligned} \Delta x^k_{R3a} = \Delta x^k_{R3s} &+ 2\sum_a \frac{\mathcal{M}_{(a)}}{d_p^2} \left\{ -d_p^2 \frac{\bar{\ell}^k_\emptyset}{2} \left(\bar{\ell}_\emptyset \cdot \tilde{v}\right) \int_{\hat{\tau}_*}^{\hat{\tau}_o} \hat{\tau} I_{0/2} \mathrm{d}\hat{\tau} - \frac{D_p^k}{2} \left(\hat{r}_p \cdot \tilde{v}\right) \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{1/2} \mathrm{d}\hat{\tau} \right. \\ &+ \left[D_v^k d_p - \frac{D_p^k}{d_p} \left(d_p \cdot \tilde{v}\right) + \frac{\bar{\ell}^k_\emptyset}{2d_p} \left(\left(\tilde{v} \times \hat{r}_p\right) \cdot \left(\hat{r}_p \times \bar{\ell}_\emptyset\right) \right) \right] \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{0/0} \mathrm{d}\hat{\tau} \\ &\left. + \left[D_p^k \left(\bar{\ell}_\emptyset \cdot \tilde{v}\right) - \frac{\bar{\ell}^k_\emptyset}{2} \left(d_p \cdot \tilde{v}\right) \right] \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{2/2} \mathrm{d}\hat{\tau} \right\} + O\left(v^2\right) + O\left(h^2\right), \end{aligned} \tag{122}$$

$$\begin{aligned} \Delta x^0_{R3b} = \bar{\ell}^0_o \Delta\hat{\tau} &+ \sum_a \frac{\mathcal{M}_{(a)}}{d_p^2} \left\{ \left(\hat{r}_p \cdot \tilde{v}\right) \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{1/1} \mathrm{d}\hat{\tau} - \left(\bar{\ell}_\emptyset \cdot \tilde{v}\right) \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{2/1} \mathrm{d}\hat{\tau} \right\} \\ &+ O\left(v^2\right) + O\left(h^2\right), \end{aligned} \tag{123}$$

$$\Delta x^k_{R3b} = \Delta x^k_{R3a} - \bar{\ell}^k_\emptyset \Delta x^0_{R3b} + O\left(v^2\right) + O\left(h^2\right), \tag{124}$$

and

$$\begin{aligned} \Delta x^0_{R4a} = \Delta x^0_{R3b} &+ 4\sum_a \frac{\mathcal{M}_{(a)}}{d_p^2} \left(\bar{\ell}_\emptyset \cdot \tilde{v}\right) \left\{ \left(\bar{\ell}_\emptyset \cdot \hat{r}_p\right) \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{1/1} \mathrm{d}\hat{\tau} - \int_{\hat{\tau}_*}^{\hat{\tau}_o} I_{2/1} \mathrm{d}\hat{\tau} \right\} \\ &+ O\left(v^2\right) + O\left(h^2\right), \end{aligned} \tag{125}$$

$$\Delta x^k_{R4a} = \Delta x^k_{R3b}, \tag{126}$$

where $\bar{\ell}^\alpha_o \equiv \bar{\ell}^\alpha(\hat{\tau}_o)$ is the line-of-sight direction at the observer used also to solve the boundary value problem [20], and the relationship $\Delta\tau = (\bar{\ell}_\emptyset \cdot \Delta\mathbf{r})$ holds, where $\Delta\tau = \hat{\tau}_o - \hat{\tau}_*$ and $\Delta r^i = r^i(\hat{\tau}_o) - r^i(\hat{\tau}_*)$.

Then, the trajectory is composed of the following terms (see the appendix), where each one of the listed solutions represents an increasing accuracy:

$$\Delta x^k_{R3s} = \bar{\ell}^k_o \Delta\hat{\tau} + 2\sum_a \mathcal{M}_{(a)} \left\{ -\frac{\bar{\ell_\emptyset}^k}{2}\left[\mathrm{Log}\left(\mathrm{r}\cdot\bar{\ell_\emptyset} + \mathrm{r}\right)\right]^{\hat{\tau}_o}_{\hat{\tau}_*} - \frac{d^k_p}{d^2_p}[r]^{\hat{\tau}_o}_{\hat{\tau}_*} \right\} + O\left(h^2\right), \tag{127}$$

$$\begin{aligned}
\Delta x^k_{R3a} = {} & \Delta x^k_{R3s} + 2\sum_a \frac{\mathcal{M}_{(a)}}{d^2_p} \left\{ d_p \left[d^k_p\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right) - \frac{\bar{\ell}^k_\emptyset}{2}\left(\hat{r}_p\cdot\tilde{v}\right)\right]\left[\arctan\left(\frac{\bar{\ell}_\emptyset\cdot r}{d_p}\right)\right]^{\hat{\tau}_o}_{\hat{\tau}_*} \right. \\
& + \left[d^k_v d_p - \frac{D^k_p}{d_p}\left(\hat{d}_p\cdot\tilde{v}\right) + \frac{\bar{\ell}^k_\emptyset}{2d_p}\left(\left(\tilde{v}\times\hat{r}_p\right)\cdot\left(\hat{r}_p\times\bar{\ell}_\emptyset\right)\right)\right] \\
& \times \left[\left(\bar{\ell}_\emptyset\cdot r\right)\arctan\left(\frac{\bar{\ell}_\emptyset\cdot r}{d_p}\right)\right]^{\hat{\tau}_o}_{\hat{\tau}_*} \\
& - \left[d^2_p d^k_v + \frac{\bar{\ell}^k_\emptyset}{2}\left(\left(\tilde{v}\times\hat{r}_p\right)\cdot\left(\hat{r}_p\times\bar{\ell}_\emptyset\right)\right) + \frac{\bar{\ell}^k_\emptyset}{2}\left(\hat{d}_p\cdot\tilde{v}\right)\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\right] \\
& \left. \times\ [\mathrm{Log}(\mathrm{r})]^{\hat{\tau}_o}_{\hat{\tau}_*} \right\} + O\left(h^2\right) + O\left(v^2\right),
\end{aligned} \tag{128}$$

$$\begin{aligned}
\Delta x^0_{R3b} = {} & \bar{\ell}^0_o\Delta\hat{\tau} + \sum_a \mathcal{M}_{(a)}\left\{ -\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\left[\mathrm{Log}\left(\mathrm{r}\cdot\bar{\ell_\emptyset} + \mathrm{r}\right)\right]^{\hat{\tau}_o}_{\hat{\tau}_*} + \frac{\left(d_p\cdot\tilde{v}\right)}{d^2_p}[r]^{\hat{\tau}_o}_{\hat{\tau}_*} \right\} \\
& + O\left(h^2\right) + O\left(v^2\right)
\end{aligned} \tag{129}$$

$$\Delta x^k_{R3b} = \Delta x^k_{R3a} - \bar{\ell}^k_\emptyset \Delta x^0_{R3b} + O\left(h^2\right) + O\left(v^2\right), \tag{130}$$

and

$$\Delta x^0_{R4a} = \Delta x^0_{R3b} - 4\sum_a \mathcal{M}_{(a)}\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\left[\mathrm{Log}\left(\mathrm{r}\cdot\bar{\ell_\emptyset} + \mathrm{r}\right)\right]^{\hat{\tau}_o}_{\hat{\tau}_*} + O\left(h^2\right) + O\left(v^2\right), \tag{131}$$

$$\Delta x^k_{R4a} = \Delta x^k_{R3b} + O\left(h^2\right) + O\left(v^2\right). \tag{132}$$

5.2. Quadrupoles

Similarly, the inclusion of the quadrupole terms, at the first order in h and $\tilde{v}$, gives additional contributions to the following trajectories (see table 1 and the appendix for the detailed expressions of each term) where, accordingly to the monopole case, each one of the listed solutions represents an increasing accuracy:

$$\begin{aligned}
\Delta x^k_{R3sQ} = {} & 6J_2R^2\frac{\mathcal{M}}{d^2_p}\left\{\left(\frac{C^k_1}{d^2_p} - \frac{2C^k_3\left(\bar{\ell}_\emptyset\cdot\hat{r}_p\right)}{3d^4_p} + \frac{2C^k_8}{3d^2_p}\right)\left[I_{1/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} - \frac{C^k_2}{d^2_p}\left[I_{2/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} - \frac{C^k_3}{3d^2_p}\right. \\
& \times\left[I_{2/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*} + C^k_4[r]^{\hat{\tau}_o}_{\hat{\tau}*} - C^k_5\left[I_{0/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} - \frac{C^k_6}{3}\left[I_{0/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*} + C^k_7\left[\hat{\tau}I_{0/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} \\
& \left. + \frac{C^k_8}{3}\left[\hat{\tau}I_{0/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*}\right\},
\end{aligned} \tag{133}$$

$$\Delta x^k_{R3aQ} = \Delta x^k_{R3a} + \Delta x^k_{R3sQ} + J_2 R^2 \frac{\mathcal{M}}{d_p^2} \Bigg\{ \Bigg(\left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right) C^k_9 + \frac{C^k_{10}}{d_p^3} - \frac{C^k_{11}\left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right)}{2d_p^3} + \frac{3C^k_{12}}{8d_p^5}$$

$$- \frac{3C^k_{13}\left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right)}{8d_p^5} + \frac{C^k_{17}}{2d_p} + \frac{3C^k_{18}}{8d_p^3} \Bigg) \left[I_{0/0}\right]^{\hat{\tau}_o}_{\hat{\tau}*} + C^k_9 \left[\hat{\tau} I_{0/0}\right]^{\hat{\tau}_o}_{\hat{\tau}} + \left(C^k_{14} - d_p C^k_9\right)$$

$$\times \left[\mathrm{Log}\ (\mathrm{r})\right]^{\hat{\tau}_o}_{\hat{\tau}*} + \frac{1}{d_p^2}\left(C^k_{11} + C^k_{15} + \left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right) C^k_{17}\right)\left[I_{2/2}\right]^{\hat{\tau}_o}_{\hat{\tau}*}$$

$$+ \frac{1}{d_p^2} \Bigg(C^k_{10} + \frac{3C^k_{12}}{4d_p^2} - \frac{3\left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right) C^k_{13}}{4d_p^2}$$

$$+ \left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right) C^k_{15} + r_p^2 C^k_{17} + \frac{3C^k_{18}}{4} \Bigg) \left[I_{1/2}\right]^{\hat{\tau}_o}_{\hat{\tau}*}$$

$$+ \frac{1}{4d_p^2}\left(C^k_{12} + \left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right) C^k_{16} + r_p^2 C^k_{18}\right)\left[I_{1/4}\right]^{\hat{\tau}_o}_{\hat{\tau}*}$$

$$- \frac{1}{4d_p^2}\left(C^k_{13} + \left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right) C^k_{18} + C^k_{16}\right)\left[I_{2/4}\right]^{\hat{\tau}_o}_{\hat{\tau}*} \Bigg\}, \tag{134}$$

$$\Delta x^0_{R3bQ} = \Delta x^0_{R3b} + J_2 R^2 \frac{\mathcal{M}}{d_p^2} \Bigg\{ \Bigg(\frac{C_{19}}{d_p^2} - \frac{2C_{21}\left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right)}{3d_p^4} + \frac{2C_{26}}{3d_p^2} \Bigg) \left[I_{1/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} - \frac{C_{21}}{d_p^2}\left[I_{2/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*}$$

$$- \Bigg(\frac{C_{21}}{3d_p^2} + \frac{2C_{26}\left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right)}{3d_p^4} \Bigg) \left[I_{2/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*} + C_{22}[r]^{\hat{\tau}_o}_{\hat{\tau}*} - C_{23}\left[I_{0/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} - \frac{C_{24}}{3}\left[I_{0/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*}$$

$$+ C_{25}\left[\hat{\tau} I_{0/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} + \frac{\hat{r}_p^2 C_{26}}{d_p^2}\left[\hat{\tau} I_{1/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*} \Bigg\}, \tag{135}$$

$$\Delta x^k_{R3bQ} = \Delta x^k_{R3b} - \bar{\ell}^k_\varnothing \Delta x^0_{R3bQ}, \tag{136}$$

and, finally,

$$\Delta x^0_{R4aQ} = \Delta x^0_{R3bQ} - 2J_2 R^2 \frac{\mathcal{M}}{d_p^2}\left(\bar{\ell}_\varnothing \cdot \tilde{v}\right) \Bigg\{ \Bigg(\frac{C_{27}}{d_p^2} - \frac{2C_{29}\left(\bar{\ell}_\varnothing \cdot \hat{r}_p\right)}{3d_p^4} + \frac{2C^k_{34}}{3d_p^2} \Bigg) \left[I_{1/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*}$$

$$- \frac{C_{28}}{d_p^2}\left[I_{2/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} - \Bigg(\frac{C_{29}}{3d_p^2} \Bigg) \left[I_{2/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*} + C_{30}[r]^{\hat{\tau}_o}_{\hat{\tau}*} + C_{31}\left[I_{0/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} - \frac{C_{32}}{3}\left[I_{0/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*}$$

$$+ C^k_{33}\left[\hat{\tau} I_{0/1}\right]^{\hat{\tau}_o}_{\hat{\tau}*} + \frac{C^k_{34}}{3}\left[\hat{\tau} I_{0/3}\right]^{\hat{\tau}_o}_{\hat{\tau}*} \Bigg\}, \tag{137}$$

$$\Delta x^k_{R4aQ} = \Delta x^k_{R3bQ}. \tag{138}$$

6. Conclusions

The Relativistic Astrometric MODel (RAMOD) is a mathematical tool conceived to model the astrometric measurements made by an observer in space. Since its original purpose was to address this problem for the ESA Gaia mission, whose final astrometric accuracy requires the physical model to be accurate at the μas level, RAMOD had to take into account the general relativistic corrections due to the bodies of the Solar System. The calculations here are performed assuming that the massive bodies move uniformly and have monopole and quadrupole structure.

Despite the apparent straightforwardness of the task and the linearity of the metric given the weak gravitational field regime inside the Solar System, the solution of the inverse ray tracing problem, which allows us to reach the aim above, is rather intractable unless treated numerically, particularly if retarded time contributions need to be accounted for [3]. As far as RAMOD is concerned, the reason lies mainly in the fact that the main unknown of the differential equations is the observed direction as projected on the rest space of the local barycentric observer and represents *locally* what the observer measures of the incoming photons in his/her gravitational environment. This aspect transforms the geodesic equation into a set of nonlinear coupled differential equations which comprises also that for the time component. The original version of the RAMOD model was therefore numerical and although successful in its applications with the inclusion of the relativistic satellite attitude [20], it was hard to control and compare with similar astrometric models even with a comprehensive error budget for stellar positions [21].

Here we present a fully analytical solution of a system of differential equations up to the ϵ^3 level everywhere in the Solar System, which is therefore able to assure a μas-level accuracy consistent with the precepts of the measurement protocol in General Relativity, and that can also be utilized under observing conditions more demanding than those of the Gaia mission.

The analytical solution is general enough to be applicable to other missions conceived to exploit photon trajectories and extends within the RAMOD formalism (since other similar solutions are already known from the literature) the analysis of the trajectory perturbations due to gravitating sources with a non-negligible quadrupole structure. While the retarded time approximation adopted here and the solution for the static cases recover the results obtained by similar astrometric models (as proved also in [11]), the solutions including the constant velocity of the source give rise to different expressions that deserve to be carefully evaluated in a dedicated work as it was done, for example, in [5] and [22], especially in consideration of the recent applications done with the Time Transfer Function approach [23]. At first glance, in fact, the presence of the time component $\bar{\ell}^0$ for the local-line-of-sight has not been contemplated in other models and the $\bar{\ell}^k$ components do not show complete coincidence. A proper comparison, both analytical and numerical, will establish the physical significance of this unexpected discrepancy as one carries on with the implementation process of the astrometric observables from which the relativistic astrometric parameters with their appropriate variance and, possibly, covariance values are deduced. This however cannot be treated in this work and deserves a separate publication as done, e.g., in Klioner and Peip [24].

Finally, we state that the main results of this article are equations (121)–(138), which represent the light trajectory through the gravitational fields of uniformly moving bodies having monopole and quadrupole structures. In the context of the application of the RAMOD model to the problem of the Gaia astrometric observations, the expressions $\Delta\bar{\ell}^\alpha$ for the light deflections are fundamental as they represent the missing ingredient for the analytical definition of the astrometric observable.

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Appendix. Expressions of the coefficients in the quadrupole deflection formulae

This appendix lists the coefficients which appear in the expression for light deflections and light trajectories in the case of an oblate body. Each coefficient depends on the linear combination of the constant vectors (with respect to the parameter $\hat{\tau}$) such as $\bar{\ell}_{\emptyset}^{k}$, D_p^k, D_s^k, etc and are denoted in order of appearance.

$$C_1^k = \frac{\bar{\ell}_{\emptyset}^{k}\left(s\cdot\bar{\ell}_{\emptyset}\right)^2}{6d_p^2}\left[2d_p^4 - 3\hat{r}_p^2 d_p^2 - 3\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^4\right];$$

$$C_2^k = \frac{\bar{\ell}_{\emptyset}^{k}}{2d_p^2}\left(s\cdot\bar{\ell}_{\emptyset}\right)^2\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\hat{r}_p^2;$$

$$C_3^k = d_p^2\left\{-D_p^k\frac{\left(s\cdot\bar{\ell}_{\emptyset}\right)^2}{2} + \frac{\bar{\ell}_{\emptyset}^{k}}{2}\left(s\cdot\bar{\ell}_{\emptyset}\right)\left[\left(s\cdot\hat{r}_p\right) - 2\left(s\cdot\bar{\ell}_{\emptyset}\right)^2\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\right]\right\};$$

$$\begin{aligned}C_4^k = {}&\frac{D_p^k}{3d_p^2}\left\{-1 + 4\frac{\left(s\cdot\hat{r}_p\right)^2}{d_p^2} + \frac{\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\left(s\cdot\bar{\ell}_{\emptyset}\right)}{d_p^2}\left[-5\left(s\cdot\hat{r}_p\right) - 3\left(\hat{r}_p\cdot d_s\right)\right]\right.\\ &\left.+\hat{r}_p^2\frac{\left(s\cdot\bar{\ell}_{\emptyset}\right)^2}{d_p^2}\right\} + \frac{\bar{\ell}_{\emptyset}^{k}}{3d_p^2}\left\{-\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right) + \frac{\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\left(s\cdot\hat{r}_p\right)}{d_p^2}\left[\left(s\cdot\hat{r}_p\right) - 3\left(\hat{r}_p\cdot d_s\right)\right]\right.\\ &\left.+\hat{r}_p^2\frac{\left(s\cdot\bar{\ell}_{\emptyset}\right)}{d_p^2}\left[-\left(s\cdot\hat{r}_p\right) + 3\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\left(s\cdot\bar{\ell}_{\emptyset}\right)\right] + \frac{\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^3\left(s\cdot\bar{\ell}_{\emptyset}\right)^2}{d_p^2}\right\} - \frac{2D_s^k}{3d_p^2}\left(\hat{r}_p\cdot d_s\right);\end{aligned}$$

$$\begin{aligned}C_5^k = {}&D_p^k\left\{-\frac{1}{2} + \frac{2\left(s\cdot\hat{r}_p\right)^2}{3d_p^2} + \frac{\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\left(s\cdot\bar{\ell}_{\emptyset}\right)}{d_p^2}\left[\frac{4}{3}\left(s\cdot\hat{r}_p\right) + \frac{\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\left(s\cdot\bar{\ell}_{\emptyset}\right)}{2}\right]\right.\\ &\left.+\frac{\left(s\cdot\bar{\ell}_{\emptyset}\right)^2 r_p^2}{6d_p^2}\right\} + \frac{\bar{\ell}_{\emptyset}^{k}}{3d_p^2}\left\{-2\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right) - \left(s\cdot\hat{r}_p\right)^2\hat{r}_p^2\frac{\left(s\cdot\bar{\ell}_{\emptyset}\right)\left(s\cdot\hat{r}_p\right)}{2}\right.\\ &\left.+\frac{\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^2\left(s\cdot\bar{\ell}_{\emptyset}\right)}{2}\left(-2\left(s\cdot\hat{r}_p\right) - \left(\hat{r}_p\cdot d_s\right)\right)\right\} - D_s^k\left(s\cdot\hat{r}_p\right);\end{aligned}$$

$$C_6^k = \frac{D_p^k}{2}\left(s \cdot \hat{r}_p\right)^2;$$

$$C_7^k = -\frac{\bar{\ell}_{\text{ø}}^k}{2} + \left(s \cdot \bar{\ell}_{\text{ø}}\right)D_s^k;$$

$$C_8^k = -\frac{D_p^k}{2}\left(s \cdot \hat{r}_p\right)^2\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right) - \frac{\bar{\ell}_{\text{ø}}^k}{2}\left\{\left(s \cdot \hat{r}_p\right)\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right) + \left(s \cdot \bar{\ell}_{\text{ø}}\right)^2\left[d_p^2 - \left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)^2\right]\right\};$$

$$\begin{aligned}
C_9^k = \frac{3D_p^k}{4d_p^3}\Bigg\{ & -\left[9\left(s \cdot \hat{r}_p\right)(s \cdot \tilde{v}) + 2\left(\hat{r}_p \cdot \tilde{v}\right)\right] \\
& + \frac{45}{2d_p^2}\left(s \cdot \hat{r}_p\right)^2\left(\hat{r}_p \cdot d_v\right) + \left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)\Bigg[9\left(s \cdot \bar{\ell}_{\text{ø}}\right)(s \cdot \tilde{v}) \\
& + 2\left(\bar{\ell}_{\text{ø}} \cdot \tilde{v}\right)\Bigg] - \frac{45}{d_p^2}\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)\left(s \cdot \bar{\ell}_{\text{ø}}\right)\left(\hat{r}_p \cdot \tilde{v}\right)\left(s \cdot \hat{r}_p\right) \\
& + \frac{9}{2d_p^2}\left(s \cdot \bar{\ell}_{\text{ø}}\right)\left[4\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)^2 + \hat{r}_p^2\right]\Bigg[2\left(s \cdot \hat{r}_p\right)\left(\bar{\ell}_{\text{ø}} \cdot \tilde{v}\right) \\
& + \left(s \cdot \bar{\ell}_{\text{ø}}\right)\left(\hat{r}_p \cdot \tilde{v}\right)\Bigg] + \frac{9}{2d_p^2}\left[2\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)^2 + 3\hat{r}_p^2\right]\left(s \cdot \bar{\ell}_{\text{ø}}\right)^2\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)\left(\ell_{\text{ø}} \cdot \tilde{v}\right)\Bigg]\Bigg\} \\
& + \frac{3D_s^k}{d_p}\Bigg\{(s \cdot \tilde{v}) - \frac{3\left(\hat{r}_p \cdot \tilde{v}\right)}{4d_p^2}\left(d_s \cdot \hat{r}_p\right) + \frac{\left(\ell_{\text{ø}} \cdot \tilde{v}\right)}{4d_p^2} \\
& \left[3\left(s \cdot \hat{r}_p\right)\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right) - \left(s \cdot \bar{\ell}_{\text{ø}}\right)\left[2\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)^2 + \hat{r}_p^2\right]\right]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{3\bar{\ell}_{\text{ø}}^k}{4d_p^3}\Bigg\{-\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)\left[9\left(s \cdot \hat{r}_p\right)(s \cdot \tilde{v}) + 2\left(\hat{r}_p \cdot \tilde{v}\right)\right] \\
& + \frac{45}{2d_p^2}\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)\left(s \cdot \hat{r}_p\right)^2\left(\hat{r}_p \cdot \tilde{v}\right) + \frac{\left[2\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)^2 + \hat{r}_p^2\right]}{3} \\
& \left[9(s \cdot \tilde{v})\left(s \cdot \bar{\ell}_{\text{ø}}\right) + 2\left(\bar{\ell}_{\text{ø}} \cdot \tilde{v}\right)\right] \\
& - \frac{9\left[4\left(\hat{r}_p \cdot \bar{\ell}_{\text{ø}}\right)^2 + \hat{r}_p^2\right]}{2d_p^2}\Bigg[\left(\bar{\ell}_{\text{ø}} \cdot \tilde{v}\right)\left(\left(s \cdot \hat{r}_p\right)^2 - \hat{r}_p^2\left(s \cdot \bar{\ell}_{\text{ø}}\right)^2\right) + 2\left(s \cdot \hat{r}_p\right)
\end{aligned}$$

$$\left(s\cdot\bar{\ell}_{\o}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\Bigg]-\frac{9\left[2\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^2+3\hat{r}_p^2\right]}{2d_p^2}\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)\left(s\cdot\bar{\ell}_{\o}\right)$$

$$\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)+\left(s\cdot\bar{\ell}_{\o}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\Bigg\}$$

$$+\frac{D_v^k}{2d_p}\left\{1+\frac{9\left(s\cdot\hat{r}_p\right)^2}{4d_p^2}+\frac{9\left(s\cdot\bar{\ell}_{\o}\right)\left(s\cdot\hat{r}_p\right)\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)}{2d_p^2}\right.$$

$$\left.+\frac{\left(s\cdot\bar{\ell}_{\o}\right)^2\left[2\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^2+\hat{r}_p^2\right]}{4d_p^2}\right\};$$

$$C_{10}^k=-27D_p^k\left\{\left(s\cdot\bar{\ell}_{\o}\right)^2\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\left[\frac{\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^4}{4d_p^2}+\frac{2\hat{r}_p^2+\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^2}{4}\right]\right\}+\frac{27\bar{\ell}_{\o}^k}{4}\Big\{\left(s\cdot\bar{\ell}_{\o}\right)$$

$$\left[\frac{\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^4}{d_p^2}+2\hat{r}_p^2+\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^2\right]\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)+\left(s\cdot\bar{\ell}_{\o}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]+\left(s\cdot\bar{\ell}_{\o}\right)^2\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)$$

$$\left.\frac{17\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)\hat{r}_p^4-28\hat{r}_p^2\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^3+16\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^5}{3d_p^2}\right\};$$

$$C_{11}^k=-D_p^k\left\{\left(s\cdot\bar{\ell}_{\o}\right)^2\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\frac{\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)\hat{r}_p^2}{2d_p^2}\right\}$$

$$+\frac{3d_p^2D_s^k}{2}\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\left(s\cdot\bar{\ell}_{\o}\right)-\frac{d_p^2D_v^k}{2}\left(s\cdot\bar{\ell}_{\o}\right)^2$$

$$+\bar{\ell}_{\o}^k\left\{d_p^2\left[\frac{9}{2}\left(s\cdot\bar{\ell}_{\o}\right)(s\cdot\tilde{v})+\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\right]\right.$$

$$+\frac{27\left(s\cdot\bar{\ell}_{\o}\right)}{4d_p^2}\Big[\hat{r}_p^2\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)\Big[2\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\left(s\cdot\hat{r}_p\right)+\left(s\cdot\bar{\ell}_{\o}\right)$$

$$\left.\left(\hat{r}_p\cdot\tilde{v}\right)\Big]+9\left(s\cdot\bar{\ell}_{\o}\right)\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\left[7\hat{r}_p^4-20\hat{r}_p^2\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^2-8\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^4\right]\Big]\right\};$$

$$C_{12}^k=9D_p^kd_p^2\hat{r}_p^2\left(s\cdot\bar{\ell}_{\o}\right)^2\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)+9\bar{\ell}_{\o}^k\Big\{\left(s\cdot\bar{\ell}_{\o}\right)^2\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)\hat{r}_p^2\left(\hat{r}_p\cdot\bar{\ell}_{\o}\right)^3-d_p^2\hat{r}_p^2\left(s\cdot\bar{\ell}_{\o}\right)$$

$$\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\o}\cdot\tilde{v}\right)+\left(s\cdot\bar{\ell}_{\o}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\Big\};$$

$$C_{13}^k = 9D_p^k d_p^2\left(s\cdot\bar{\ell}_\emptyset\right)\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)+\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)+3\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\left(s\cdot\bar{\ell}_\emptyset\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right]$$
$$+9\bar{\ell}_\emptyset^k\left\{\left(s\cdot\bar{\ell}_\emptyset\right)^2\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\left[\hat{r}_p^4-5d_p^2\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\right]-3d_p^2\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right.\right.$$
$$\left.\left.+\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]-\left(s\cdot\hat{r}_p\right)d_p^2\left[\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)+2\left(\hat{r}_p\cdot\tilde{v}\right)\left(s\cdot\bar{\ell}_\emptyset\right)\right]\right\};$$

$$C_{14}^k = \frac{3D_p^k}{2d_p^2}\left\{9\left(s\cdot\hat{r}_p\right)(s\cdot\tilde{v})\right.$$
$$+2\left(\hat{r}_p\cdot\tilde{v}\right)+\frac{45}{2d_p^2}\left(s\cdot\hat{r}_p\right)^2\left(\hat{r}_p\cdot d_v\right)+\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\left[9\left(s\cdot\bar{\ell}_\emptyset\right)(s\cdot\tilde{v})\right.$$
$$\left.+2\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right]-\frac{45}{d_p^2}\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\left(s\cdot\hat{r}_p\right)$$
$$+\frac{9}{2d_p^2}\left[4\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)^2+\hat{r}_p^2-\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\right.$$
$$\left.\left.\left(\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)^2-\frac{3\hat{r}_p^2}{2}\right)\right]\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)+\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\right\}$$
$$+3D_s^k\left\{(s\cdot\tilde{v})+\frac{\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)}{d_p^2}\right.$$
$$\left.\left[\left(\hat{r}_p\cdot\tilde{v}\right)\left(s\cdot\bar{\ell}_\emptyset\right)+\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right]-\frac{\left(\ell_\emptyset\cdot\tilde{v}\right)}{4d_p^2}\left[2\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)^2+\hat{r}_p^2\right]\left(s\cdot\bar{\ell}_\emptyset\right)\right\}$$
$$+\frac{3\bar{\ell}_\emptyset^k}{4d_p^2}\left\{-\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\right.$$
$$\left[9\left(s\cdot\hat{r}_p\right)(s\cdot\tilde{v})+2\left(\hat{r}_p\cdot\tilde{v}\right)\right]+\frac{45}{2d_p^2}\left(r_p\cdot\bar{\ell}_\emptyset\right)\left(s\cdot\hat{r}_p\right)^2\left(\hat{r}_p\cdot\tilde{v}\right)$$
$$+\left[2\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)^2+\hat{r}_p^2\right]\left[\frac{3}{2}(s\cdot\tilde{v})\left(s\cdot\bar{\ell}_\emptyset\right)\right.$$
$$\left.+\frac{1}{3}\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\right]-\frac{9\left[4\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)^2+\hat{r}_p^2\right]}{2d_p^2}\left[\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\left(\left(s\cdot\hat{r}_p\right)^2-\hat{r}_p^2\left(s\cdot\bar{\ell}_\emptyset\right)^2\right)\right.$$
$$\left.+2\left(s\cdot\hat{r}_p\right)\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]$$
$$\left.-\frac{9\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)\left(s\cdot\bar{\ell}_\emptyset\right)\left[2\left(\hat{r}_p\cdot\bar{\ell}_\emptyset\right)^2+3\hat{r}_p^2\right]}{2d_p^2}\left[2\left(\bar{\ell}_\emptyset\cdot\tilde{v}\right)\left(s\cdot\hat{r}_p\right)+\left(s\cdot\bar{\ell}_\emptyset\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\right\}$$

$$+ \frac{D_v^k}{2}\left\{1 - \frac{\left(s\cdot\hat{r}_p\right)^2}{d_p^2} + \frac{9\left(s\cdot\bar{\ell}_\text{ø}\right)\left(s\cdot\hat{r}_p\right)\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)}{2d_p^2}\right.$$

$$\left. + \frac{\left(s\cdot\bar{\ell}_\text{ø}\right)^2\left[2\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)^2 + \hat{r}_p^2\right]}{4d_p^2}\right\};$$

$$C_{15}^k = \frac{D_p^k}{2}\left\{\left(s\cdot\hat{r}_p\right)\left(s\cdot\tilde{v}\right) + \frac{2}{9}\left(\hat{r}_p\cdot\tilde{v}\right) + \frac{5}{d_p^2}\left(s\cdot\hat{r}_p\right)\left[\left(s\cdot\hat{r}_p\right)\left(\hat{r}_p\cdot\tilde{v}\right) - \left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)\left(\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right)\right.\right.\right.$$

$$\left.\left.\left(s\cdot\hat{r}_p\right) + 2\left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right)\right]$$

$$\left. + 3\left(s\cdot\bar{\ell}_\text{ø}\right)\frac{4\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)^2 + \hat{r}_p^2}{2d_p^2}\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right) + \left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\right\}$$

$$- \frac{3D_v^k}{4}\left(s\cdot\hat{r}_p\right)^2 + \frac{3D_s^k}{2}\left(s\cdot\hat{r}_p\right)\left(\hat{r}_p\cdot\tilde{v}\right) + \frac{9\bar{\ell}_\text{ø}^k}{4d_p^2}\left\{5\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)\left(s\cdot\hat{r}_p\right)^2\left(\hat{r}_p\cdot\tilde{v}\right)\right.$$

$$\left. - 3\left(s\cdot\hat{r}_p\right)\left[4\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)^2 + \hat{r}_p^2\right]\left[\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right)\left(s\cdot\hat{r}_p\right) + 2\left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\right\};$$

$$C_{16}^k = 9D_p^k\left(s\cdot\hat{r}_p\right)^2\left(\hat{r}_p\cdot\tilde{v}\right);$$

$$C_{17}^k = \frac{D_p^k}{2}\left\{\left[9\left(s\cdot\bar{\ell}_\text{ø}\right)\left(s\cdot\tilde{v}\right) + 2\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right)\right]\right\}$$

$$+ \frac{\bar{\ell}_\text{ø}^k}{2}\left\{\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)\left[9\left(s\cdot\bar{\ell}_\text{ø}\right)\left(s\cdot\tilde{v}\right) + 2\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right)\right]\right.$$

$$\left. - \left[9\left(s\cdot\hat{r}_p\right)\left(s\cdot\tilde{v}\right) + 2\left(\hat{r}_p\cdot\tilde{v}\right)\right]\right\}$$

$$+ \frac{3D_s^k}{2}\left[\left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\tilde{v}\right) + \left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right)\left(\left(s\cdot\hat{r}_p\right)\right.\right.$$

$$\left.\left. - \left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)\right)\right] + \frac{3D_v^k}{4}\left(s\cdot\bar{\ell}_\text{ø}\right)^2\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right);$$

$$C_{18}^k = -9D_p^k\left\{3\left(s\cdot\hat{r}_p\right)\left[\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right)\right.\right.$$

$$\left. + 2\left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right] - \left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right)\right.$$

$$\left.\left. + \left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right] - 3\left(s\cdot\bar{\ell}_\text{ø}\right)^2\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right)\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)^2\right\}$$

$$- 9\bar{\ell}_\text{ø}^k\left\{3\left(s\cdot\hat{r}_p\right)^2\left(\hat{r}_p\cdot\tilde{v}\right) + \left(s\cdot\hat{r}_p\right)\left(\hat{r}_p\cdot\bar{\ell}_\text{ø}\right)\right.$$

$$\left[\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_\text{ø}\cdot\tilde{v}\right) + 2\left(s\cdot\bar{\ell}_\text{ø}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]$$

$$
\begin{aligned}
&+\left(s\cdot\bar{\ell}_{\emptyset}\right)\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^2\Big[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)+\left(s\cdot\bar{\ell}_{\emptyset}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\\
&-d_p^2\left(s\cdot\bar{\ell}_{\emptyset}\right)^2\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\Big]\Big\};
\end{aligned}
$$

$$
C_{19}=-\frac{1}{2d_p^2}\left\{\left[2d_p^4-3\hat{r}_p^2d_p^2+3\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^4\right)\right]\left(s\cdot\bar{\ell}_{\emptyset}\right)^2\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\right\};
$$

$$
C_{20}=-\frac{3\hat{r}_p^2}{2d_p^2}\left(s\cdot\bar{\ell}_{\emptyset}\right)^2\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right);
$$

$$
C_{21}=3d_p^2\left(s\cdot\bar{\ell}_{\emptyset}\right)\left[\frac{1}{2}\left(\hat{r}_p\cdot\tilde{v}\right)\left(s\cdot\bar{\ell}_{\emptyset}\right)+\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\left(\hat{r}_p\cdot d_s\right)\right];
$$

$$
\begin{aligned}
C_{22}=\frac{1}{2d_p^2}\Big\{&\left(\hat{r}_p\cdot d_v\right)-\frac{4}{d_p^2}\left(\hat{r}_p\cdot\tilde{v}\right)\left(s\cdot\hat{r}_p\right)^2\\
&+\frac{4}{d_p^2}\left(s\cdot\hat{r}_p\right)\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\Big[\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\\
&+2\left(s\cdot\bar{\ell}_{\emptyset}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\Big]-\frac{\left(s\cdot\bar{\ell}_{\emptyset}\right)}{d_p^2}\left[\hat{r}_p^2+3\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^2\right]\\
&\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)+\left(s\cdot\bar{\ell}_{\emptyset}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\Big\};
\end{aligned}
$$

$$
\begin{aligned}
C_{23}=&\frac{1}{2d_p^2}\Big\{3\left(\hat{r}_p\cdot\tilde{v}\right)d_p^2-4\left(\hat{r}_p\cdot\tilde{v}\right)\left(s\cdot\hat{r}_p\right)^2\\
&+4\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\left(s\cdot\hat{r}_p\right)\Big[\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)+2\left(s\cdot\bar{\ell}_{\emptyset}\right)\\
&\left(\hat{r}_p\cdot\tilde{v}\right)\Big]+\left(s\cdot\bar{\ell}_{\emptyset}\right)\left[\hat{r}_p^2+3\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^2\right]\left[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)+\left(\hat{r}_p\cdot\tilde{v}\right)\left(s\cdot\bar{\ell}_{\emptyset}\right)\right]\\
&+\left(s\cdot\bar{\ell}_{\emptyset}\right)^2\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^3\Big\};
\end{aligned}
$$

$$
C_{24}=\left(s\cdot\hat{r}_p\right)^2\left(\hat{r}_p\cdot\tilde{v}\right);
$$

$$
C_{25}=-\frac{\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)}{2};
$$

$$
\begin{aligned}
C_{26}=-\frac{3}{2}\Big\{&\left(s\cdot\hat{r}_p\right)\left[\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)+2\left(s\cdot\bar{\ell}_{\emptyset}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\right]\\
&+\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)\left(s\cdot\bar{\ell}_{\emptyset}\right)\Big[2\left(s\cdot\hat{r}_p\right)\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\\
&+\left(s\cdot\bar{\ell}_{\emptyset}\right)\left(\hat{r}_p\cdot\tilde{v}\right)\Big]+\left(s\cdot\bar{\ell}_{\emptyset}\right)^2\left(\bar{\ell}_{\emptyset}\cdot\tilde{v}\right)\left[d_p^2-\left(\hat{r}_p\cdot\bar{\ell}_{\emptyset}\right)^2\right]\Big\};
\end{aligned}
$$

$$C_{27} = \frac{\left(s \cdot \bar{\ell}_{\o}\right)^2}{3}\left[2d_p^4 - 3\hat{r}_p^2 d_p^2 - 3\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right)^4\right];$$

$$C_{28} = -\frac{\hat{r}_p^2}{d_p^2}\left(s \cdot \bar{\ell}_{\o}\right)^2\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right);$$

$$C_{29} = d_p^2\left(s \cdot \bar{\ell}_{\o}\right)\left[\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right)\left(s \cdot \bar{\ell}_{\o}\right) + 2\left(s \cdot \hat{r}_p\right)\right];$$

$$C_{30} = -\frac{4}{d_p^2}\left(s \cdot \bar{\ell}_{\o}\right)\left(s \cdot d_p\right) - \frac{2}{3d_p^4}\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right)\left[2\left(s \cdot \hat{r}_p\right)\left(\left(s \cdot \bar{\ell}_{\o}\right)\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right) + 2\left(s \cdot \hat{r}_p\right)\right)\right.$$
$$\left. - \left(s \cdot \bar{\ell}_{\o}\right)^2\left[\hat{r}_p^2 + \left(\hat{r}_p \cdot \bar{\ell}_{\o}\right)^2\right]\right];$$

$$C_{31} = -\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right) - 6\left(\hat{r}_p \cdot s\right)\left(s \cdot \bar{\ell}_{\o}\right)$$
$$+ \frac{4}{3d_p^2}\left(\hat{r}_p \cdot s\right)\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right)\left[2\left(s \cdot \bar{\ell}_{\o}\right)\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right) + 3\left(\hat{r}_p \cdot s\right)\right]$$
$$+ \frac{\hat{r}_p^2 + 3\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right)^2}{3d_p^2}\left(s \cdot \bar{\ell}_{\o}\right)\left[\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right)\left(s \cdot \bar{\ell}_{\o}\right) + 2\left(\hat{r}_p \cdot s\right)\right]$$
$$- \frac{\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right)^3\left(s \cdot \bar{\ell}_{\o}\right)^2}{3d_p^2};$$

$$C_{32} = \left(s \cdot \hat{r}_p\right)^2\left(\hat{r}_p \cdot \bar{\ell}_{\o}\right);$$

$$C_{33} = \frac{1}{3}\left[1 + 6\left(s \cdot \bar{\ell}_{\o}\right)^2\right].$$

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