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Geometrical tools for the analysis of X-ray polarimetric signals

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ABSTRACT

X-ray polarimetric measurements are based on the study of distributions of the directions of scattered photons or photoelectrons and the search of a sinusoidal modulation with a period of π. We present a new simple tool based on a scatter plot of the modulation curve in which the counts in each angular bin are reported after a shifting by 1/4 of the period. The sinusoidal pattern is thus transformed in a circular plot whose radius is equal to the amplitude of the modulation, while for a not polarized radiation the scatter plot is reduced to a random point distribution centred at the mean frequency value. The advantage of this tool is that one can easily evaluate the statistical significance of the polarimetric detection and can obtain useful information on the quality of the measurement.

Keywords: Polarization, X-rays, Geometrical tool

1. POLARIZATION DATA

For each detected photon the information on its linear polarization is given by the azimuthal angle φ measured by the direction of a scattered photon (for scattering polarimeters) or of the initial direction of the photoelectron trajectory (for photoelectric polarimeters). The detection of a polarization is therefore obtained from the angular distribution of the φ values of a number N of detected photons (modulation curve) and particularly from the amplitude of a sinusoidal modulation with a period of π (or 180°). In the following we will refer to this histogram as the data set $\{n\}$, its total bin number is M and a number of events in the k-th bin indicated by n_k $(1 \leq k \leq M)$. As it will be clear in the following it is better that M would be a multiple of 8 (in the following we will assume bin numbers not lower than 48). We apply our methods to a polarization angle distribution as the one shown in Fig.1, obtained in laboratory tests using a totally polarized source.

For a total number of counts N one obviously has

$$\langle n \rangle = N/M \tag{1}$$

2. THE METHOD OF THE CIRCULAR PLOT

Our tool for measuring the polarization parameters is based on the scatter plot of the φ histogram in the (ξ, η) space, where these quantities are the histogram values with the latter one shifted by $M/8$ bins. Consider the distribution of M points having the coordinates $(\xi_k = n_k, \eta_k = n_{k+M/8})$, $(1 \leq k \leq 7M/8)$ and circularly completed with the $M/8$ points $(\xi_k = n_k, \eta_k = n_{k-7M/8})$, $(7M/8 + 1 \leq k \leq M)$. If the counts are distributed with a $\cos(2\varphi)$ modulation the difference of $M/8$ points corresponds to 1/4 of the period and the points in the

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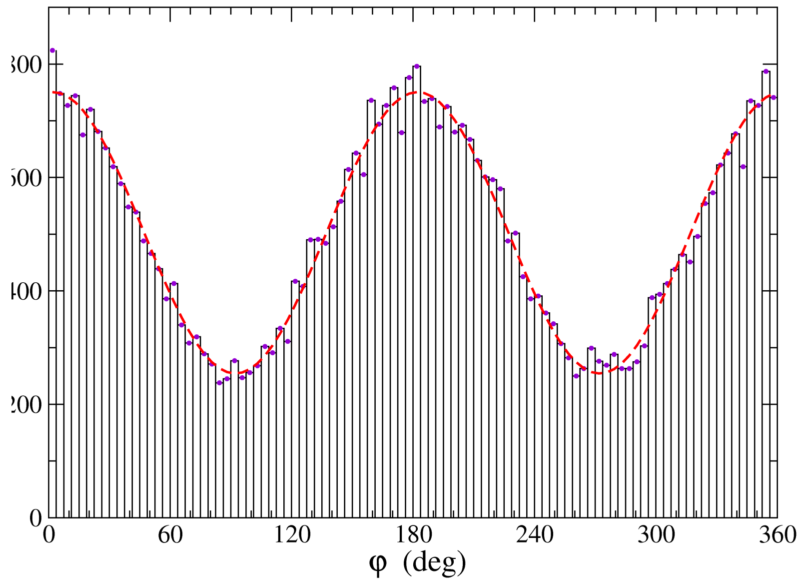

Figure 1. Histogram of the angular distribution of the initial direction of electrons due to a polarized X-ray flux. The dashed red curve is the best fit modulation.

(ξ, η) space will be distributed in a circular pattern having a radius equal to the modulation amplitude. For each point we can compute the radius R_k^2 of this circle

$$R_k^2 = (n_k - \langle n \rangle)^2 + (n_{k+M/8} - \langle n \rangle)^2 = (\xi_k - N/M)^2 + (\eta_k - N/M)^2 \tag{2}$$

The plot of R_k^2 against the bin angle φ_k is expected to be constant and the ratio between the standard deviation of data with respect to its mean value and the radius itself is another estimator of the S/N ratio. In Fig. 2 (left panel) it is reported the circular polarization plot for the data set of Fig. 1: the radius equal to square root of the mean of the $\{R_k^2\}$ values is $R_q = 249.20$.

Note that in the circular plot each value n_k is used two times: first for computing η and after for ξ; thus the M values distributed in the two circles are not entirely independent. For instance, a large positive fluctuation in one of the $\{n\}$ values would correspond to two values of R_k in excess, the former one in the η component and, after $M/8$ points, in the ξ component.

The polarization degree p can be estimated from the ratio

$$p = R_q/[\mu\langle n \rangle] = R_q/[\mu(N/M)] \tag{3}$$

where μ is the *modulation factor* of the polarimeter. The polarization angle ψ is estimated by means of the simple formula (written in degrees):

$$\psi = (1/2)(\langle \beta \rangle - 360^\circ) \tag{4}$$

where $\langle \beta \rangle$ is the mean value of the angles β_k, which are those between the ξ axis and the vector radius of the circular plot and whose trigonometric tangent is tg $\beta_k = (\eta_k - N/M)/(\xi_k - N/M)$. This mean must be computed over M values, corresponding to two complete rounds in the circle with increasing angles up to $360^\circ \times 2$, and thus 360° is equal to the mean expected increase of the angle. If the mean β would be computed over a single round ($M/2$ values) the constant 360° must be decreased to 180°.

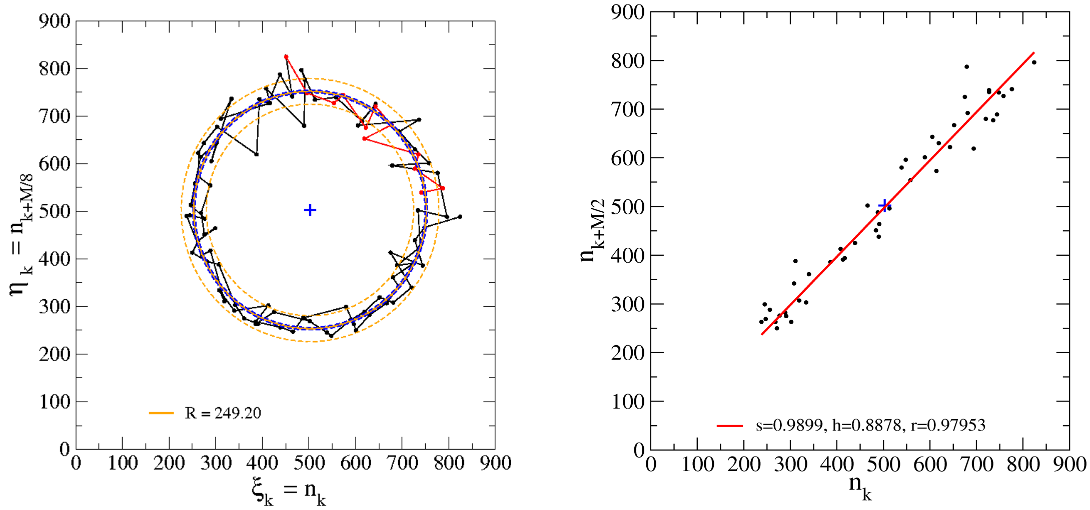

Figure 2. Left panel: scatter plot of the data of Fig. 1 exhibiting the circular pattern due to the shift of $M/8$ data shift in a 96 bin angular distribution. Red points are those of the circular completion. The central orange circle has a radius equal to the average radial distance from the centre marked by the blue cross, while the inner and outer ones correspond to $\pm 1\sigma$ of the data and of the mean. Right panel: scatter plot exhibiting the linear correlation between the first and second sections of a 96 bin angular distribution in Fig. 1. The red solid line is the linear best fit to the data.

2.1 Variance of the parameters of a circular plot

It is interesting to study how the variance of the radius R is related to the variance of $\{n\}$.

$$R_k^2 = (n_k - \langle n \rangle)^2 + (n_{k+M/8} - \langle n \rangle)^2 \tag{5}$$

and

$$\langle R^2 \rangle = \frac{1}{M} \sum_{k=1}^{M} \left[(n_k - \langle n \rangle)^2 + (n_{k+M/8} - \langle n \rangle)^2 \right] = 2\sigma_n^2 \tag{6}$$

that for not polarized data is equal the Poissonian variance N/M. One has also

$$\Delta R_k = \sqrt{(n_k - \langle n \rangle)^2 + (n_{k+M/8} - \langle n \rangle)^2} - \langle R \rangle \tag{7}$$

and

$$\sigma_R^2 = \langle (\Delta R)^2 \rangle = \frac{1}{M} \sum_{k=1}^{M} [R_k^2 + \langle R \rangle^2 - 2\langle R \rangle R_k] = \langle R^2 \rangle - \langle R \rangle^2 = 2\sigma_n^2 - \langle R \rangle^2 \qquad . \tag{8}$$

The standard deviation of $\langle R \rangle$ then will be

$$\sigma_{\langle R \rangle} = \frac{1}{\sqrt{M}}\, \sigma_R = \frac{1}{\sqrt{M}} \sqrt{2\sigma_n^2 - \langle R \rangle^2} \qquad . \tag{9}$$

For the polarization degree we have:

$$\sigma_p = \sigma_{\langle R \rangle} / [\mu(N/M)] = \frac{\sqrt{M}}{\mu N} \sqrt{2\sigma_n^2 - \langle R \rangle^2} \qquad . \tag{10}$$

The variance of the polarization angle can be computed by that of the residuals of β with respect to the linear fit $\sigma_{\beta r}$ and with the approximation $M \gg 1$ one obtains:

$$\sigma_\psi = \frac{\sigma_{\beta r}}{\sqrt{M\ (1 - \langle k \rangle^2/\langle k^2 \rangle)}} \approx 2\sigma_{\beta r}/\sqrt{M} \qquad . \tag{11}$$

A further comment is on the evaluation of the S/N ratio. In fact, this ratio should be given by $\langle R \rangle/\sigma_R$ that measures how large is the distance to the origin in units of the standard deviation of R. The use of $\sigma_{\langle R \rangle}$ instead of σ_R is not correct because it does not take into account the actual dispersion of R values around its mean and will give a meaningless high S/N value.

3. THE METHOD OF THE LINEAR CORRELATION PLOT

We create a scatter plot with $M/2$ points having coordinates $(n_k, n_{k+M/2})$ $(1 \le k \le M/2)$. If a polarization modulated signal is present these points will be distributed along a straight segment with an inclination of 45° and with a length equal to the double of the modulation amplitude. The linear correlation coefficient r can be considered a reliable estimator of the S/N ratio and the significance of the polarization measurement is obtained from that having the corresponding r with $M/2$ degrees of freedom. The linear plot for the same data set in Fig. 1 is given in Fig. 2 (right panel), where the values of the linear correlation coefficient and of the best fit line:

$$n_{k+M/2} = h + s\ n_k \tag{12}$$

are also reported.

In the case of a non polarized radiation we expect no correlation between the phase bins and in the plot the corresponding points will be randomly distributed around the mean value, with a linear correlation coefficient close to 0.

It is well known from the correlation theory that r^2 is the fraction of the variance *explained* by the linear regression and that $1 - r^2$ is the *residual* variance due to the noise; thus the simplest way to evaluate the S/N ratio of the polarization is that of computing $r^2/(1 - r^2)$. We stress that this simple formula is independent of any assumption of the nature of the noise and particularly if it is only due to the Poisson statistics or to the occurrence of possible systematic effects.

4. NUMERICAL SIMULATIONS

We used numerical simulations of polarization angular distributions with a Poissonian statistics to investigate some statistical properties of the proposed tools. Fig. 3 illustrate the same plots of Fig. 2 but for simulated data for an unpolarized (black points) and 40% polarized signals with histograms of 192 angular bins. Points of the former do not track any circle and appear randomly distributed around the mean value of counts: note that the mean (yellow) circle corresponds to a non zero polarization, but its radius equal to 67.3 is lower than 2 standard deviation (orange outer and inner circles correspond to $\pm 1\sigma$ of the radius values). In the latter case the mean radius of the circular distribution is more than 21 standard deviations and the estimated value of the polarization degree is 0.401 ± 0.019.

The capability of our tools for measuring the polarization is also shown by the linear correlation plot in the right panel of Fig. 3: here the unpolarized data have a correlation coefficient $r = -0.005$, while in the case of the polarized data it results $r = 0.9938$, and therefore only the ≈1.2% of the total variance is due to statistical noise.

It is also interesting to derive the histograms of the radial distances (see Fig. 4): that of the polarized data exhibits a Gaussian-like shape while that of the unpolarized distances is much close to the zero and is characterized by the asymmetric Rayleigh profile.

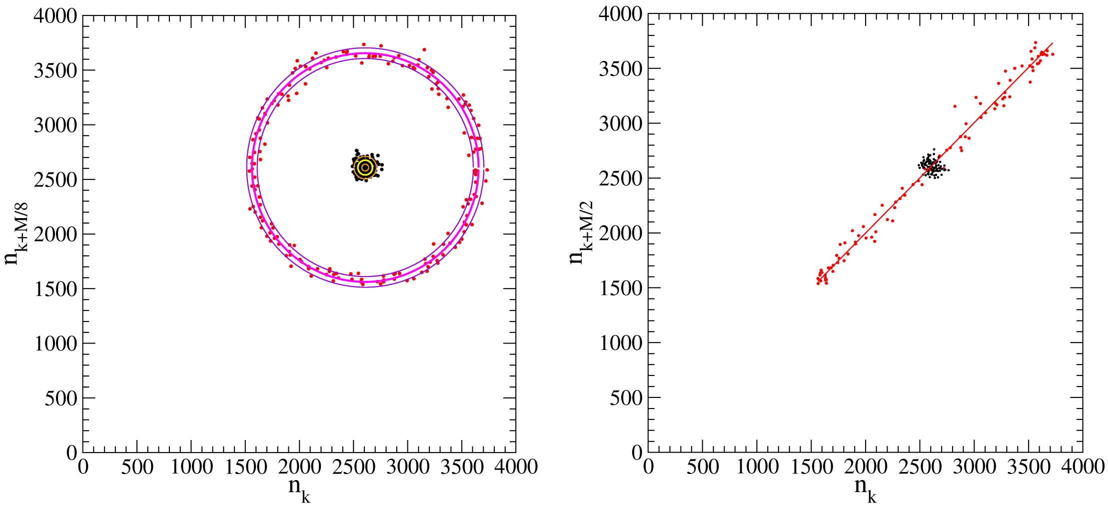

Figure 3. Left panel: circular plots of simulated polarization data for a 192 bin angular distributions of simulated unpolarized (black dots) and a 40% polarized signals (red dots). The orange and magenta circles have a radius equal to the average radii and the outer and inner circles correspond to $\pm 1\sigma$. Right panel: linear scatter plot correlation for the same simulated data. The red line is the best fit of polarized data.

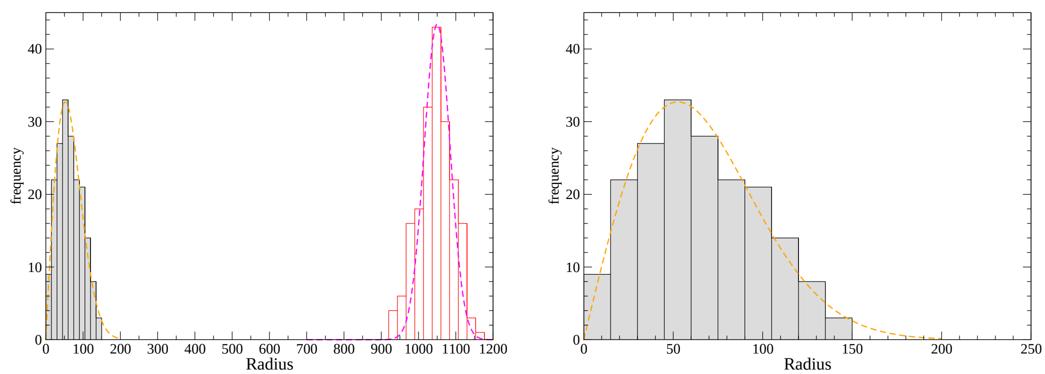

Figure 4. Left panel: Histograms of the radial values for the two simulated data set of Fig.3. The histogram of the unpolarized signal (the gray one on the left) appears well described by a Rayleigh function (orange dashed line, see the detail in the right panel), while that of the polarized signal (the red one on the right side) is more symmetric and rather similar to a Gaussian function (magenta dashed line).

5. THE STOKES' PARAMETERS

The most usual method to estimate the polarization parameter is that of computing the Stokes' parameters, that in the case of a linear polarization measure by means of an angular histogram are (Kislat et al. 2015):

$$Q = \sum_{k=1}^{M} n_k \ \cos(2\varphi_k) \quad , \quad U = \sum_{k=1}^{M} n_k \ \sin(2\varphi_k) \quad . \tag{13}$$

It is clear from the scalar products of these formulae that they correspond to the calculation of the a second harmonic cosine and sine components of the Fourier series. It is therefore very important to properly take into account the amplitude distribution over all the other frequencies due to noise fluctuations to evaluate the true significance. The estimated of the statistical uncertainties of the polarization parameters requires an accurate treatment as demonstrated in the paper by Kislat et al. (2015). In our method the statistical properties of data are entirely preserved.

6. CONCLUSION

We proposed two simple tools for the measuring the polarization parameters based on correlations between pairs of bins in the modulation curve having selected shifts. We verified with laboratory data and many numerical simulations that the results of the circular plot method are fully consistent with those obtained by Stokes' parameters, indicating that our estimator is unbiased. Some statistical properties can be demonstrated in a straightforward way: for instance, for a non polarized signal the radial distribution of points in the circular plot is the Rice distribution because their distances from the centre are those expected by a random gaussian process (see for instance Elsner et al. 2012). No particular on the origin of the noise (for instance, due to Poissonian fluctuations) is necessary for computing the S/N ratio and variance of the estimated parameters' values. Moreover, these tools are well suited for the analysis of data obtained by scattering polarimeters where the absorber is segmented into several scintillator elements (see, for instance, the COMPASS project, Del Monte et al. 2016) and the angular distribution is directly obtained as an histogram.

A more complete statistical analysis and the other refined algorithms for measuring the polarization parameters, particularly for low polarization degree, will be discussed in a more extended work.

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