The Astrophysical Journal, 837:158 (8pp), 2017 March 10 https://doi.org/10.3847/1538-4357/aa633a


# On Weibull's Spectrum of Nonrelativistic Energetic Particles at IP Shocks: Observations and Theoretical Interpretation

G. Pallocchia, M. Laurenza, and G. Consolini
INAF—Istituto di Astrofisica e Planetologia Spaziali, Via Fosso del Cavaliere 100, I-00133 Roma, Italy
*Received 2016 June 22; revised 2017 February 24; accepted 2017 February 24; published 2017 March 15*

## Abstract

Some interplanetary shocks are associated with short-term and sharp particle flux enhancements near the shock front. Such intensity enhancements, known as shock-spike events (SSEs), represent a class of relatively energetic phenomena as they may extend to energies of some tens of MeV or even beyond. Here we present an SSE case study in order to shed light on the nature of the particle acceleration involved in this kind of event. Our observations refer to an SSE registered on 2011 October 3 at 22:23 UT, by *STEREO B* instrumentation when, at a heliocentric distance of 1.08 au, the spacecraft was swept by a perpendicular shock moving away from the Sun. The main finding from the data analysis is that a Weibull distribution represents a good fitting function to the measured particle spectrum over the energy range from 0.1 to 30 MeV. To interpret such an observational result, we provide a theoretical derivation of the Weibull spectrum in the framework of the acceleration by "killed" stochastic processes exhibiting power-law growth in time of the velocity expectation, such as the classical Fermi process. We find an overall coherence between the experimental values of the Weibull spectrum parameters and their physical meaning within the above scenario. Hence, our approach based on the Weibull distribution proves to be useful for understanding SSEs. With regard to the present event, we also provide an alternative explanation of the Weibull spectrum in terms of shock-surfing acceleration.

*Key words:* acceleration of particles – magnetohydrodynamics (MHD) – plasmas – shock waves – solar wind – turbulence

## 1. Introduction

One of the most intriguing and unsolved problems of astrophysics is the particle acceleration to high energies in space plasmas. Fermi's acceleration mechanism (Fermi 1949) is a theoretical tool extensively used in astrophysical contexts and also in other research fields such as plasma physics (Michalek et al. 1999) and in the theory of dynamical systems (Zaslavskii & Chirikov 1965; Lichtenberg & Lieberman 1991). The first-order acceleration, a variant of the original Fermi mechanism, constitutes the basics for the diffusive shock acceleration (DSA) (e.g., Krymskii 1977; Blandford & Ostriker 1978) wherein a particle, repeatedly scattered across the shock front, gains energy through head-on collisions against the converging downstream and upstream plasma irregularities. The DSA naturally produces a power-law energy spectrum that is accepted to explain the observed cosmic-ray spectrum up to about $10^{15}$ eV (Blandford & Eichler 1987). Moreover, in the inner heliosphere, many events of particle acceleration at the shock have been reported in which the slopes of the particle power-law spectra (in the range from a few tens up to a few hundreds of keV) have values consistent with those predicted by DSA (Giacalone 2012; Neergaard Parker & Zank 2012). Hence, the DSA approach has received the most attention to interpret particle acceleration at shock waves, although it does not fully address several aspects of the phenomenon. For instance, the expected relationship between the power-law spectral index and the shock compression ratio at interplanetary shocks can be loose when checked through observations (van Nes et al. 1984; Fisk & Gloeckler 2012). Moreover, observations of solar energetic particle (SEP) events have shown that the predicted power law is valid on a limited energy interval (e.g., Mewaldt et al. 2005) below a characteristic energy where the spectrum has a rollover. An exponential decay was only heuristically introduced to take into account this feature (Ellison & Ramaty 1985), where the rollover energy is supposed to depend on several parameters related to the interplanetary shock (Lee et al. 2012). In addition, DSA is conceptually difficult at quasi-perpendicular shocks, due to the high particle energies requested for the injection in the acceleration process and also to the insufficient upstream wave energy density accountable for particle scattering (Zank et al. 2006; Neergaard Parker et al. 2014). A combination of DSA with magnetic-island-reconnection-related processes has been recently proposed to overcome some inconsistencies between the standard DSA predictions and observations of particle acceleration at interplanetary shocks (Zank et al. 2015).

Shock-surfing and shock-drift acceleration (hereafter SSA and SDA) are two non-Fermi mechanisms that can provide particle pre-acceleration to reach the energy threshold required to start DSA at quasi-perpendicular shocks. In SSA and DSA particle energization is caused by electric fields. In SSA (Sagdeev 1966; Ohsawa 1986; Lee et al. 1996) some particles are trapped upstream of the shock (i.e., they surf the wave) and accelerated, along the shock front and perpendicularly to the magnetic field, through the combined action of the electrostatic potential gradient (normal to the wavefront) and the Lorentz force. Particles are de-trapped when reach enough energy to breach the electrostatic barrier and escape downstream. Vice versa, in SDA (Pesses et al. 1982; Decker & Vlahos 1985) the gradient of the magnetic field at the shock ramp causes a drift of particle guiding centers so that the particles can gain energy experiencing a displacement along the direction of the convective electric field as they proceed downstream.

Pump acceleration (e.g., Fisk & Gloeckler 2012, 2014) is a further non-Fermi mechanism that may be invoked in disparate plasma conditions into the heliosphere. According to the pump

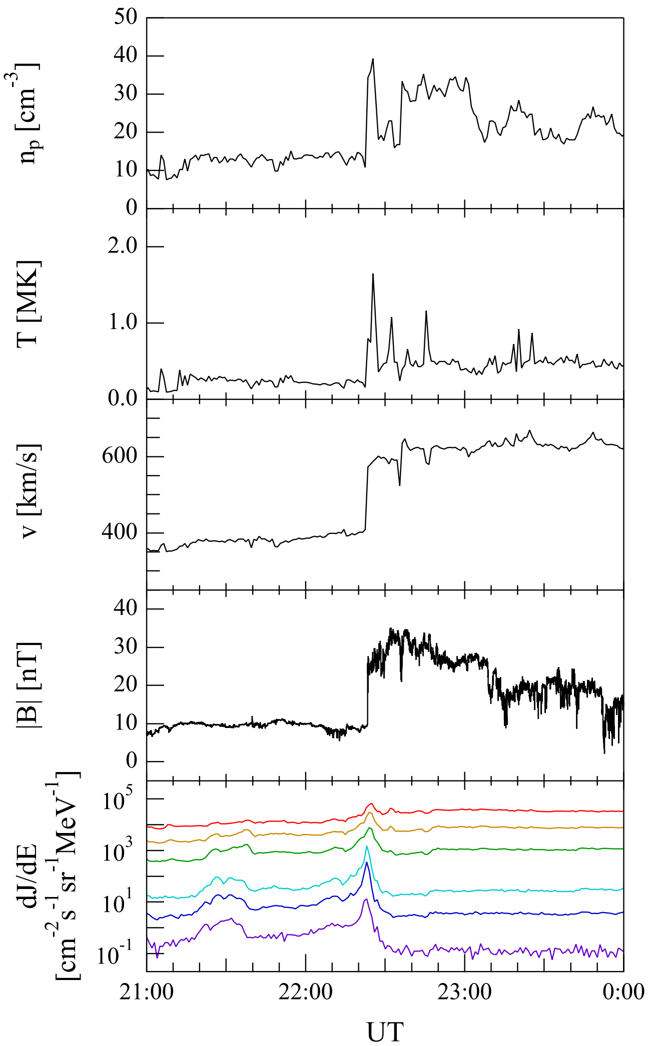


**Figure 1.** Time history of solar wind plasma parameters and energetic particle fluxes as recorded by *STEREO B* s/c between 21:00 UT and 24:00 UT on 2011 October 3. From top to bottom: proton density $n_p$ and temperature $T$, bulk speed $v$, magnetic field magnitude, and the proton differential fluxes for a selected number of energy channels ($E \sim 0.53$, 1.05, 2.10, 4.74, 6.93, and 10.95 MeV from top to bottom).

mechanism, in a volume plasma experiencing a sequence of adiabatic compressions and expansions, the energy of particles of the core population is pumped up to the suprathermal tail within an overall process of the redistribution of the particle energy content. The spectrum determined by the pump acceleration is always the same, independently from the particular physical conditions in which the mechanism is considered. Such a spectrum, known as the "common spectrum," is characterized by a distribution function having, at low speeds, a power-law part with spectral index −5 and an exponential rollover at higher speeds. In the inner heliosphere the common spectrum has been observed at quite low energies, mostly below 100 keV/nucleon, in many events of shock acceleration recorded at the Lagrangian point $L_1$ by the *ACE* spacecraft during the year 2001 (Fisk & Gloeckler 2014).

On the other hand, stochastic acceleration (SA), also called second-order acceleration and based on the original Fermi mechanism, is characterized by an average energy gain due to the particle interaction with randomly moving magnetized clouds or turbulent fluctuations. The SA has been proposed to play a dominant role in many other astrophysical environments where particles can be accelerated in a bounded space region such as radio galaxies (Eilek 1979), solar flares (Petrosian & Liu 2004), the interstellar medium (Seo & Ptuskin 1994), and supernova remnants (Scott & Chevalier 1975). A few theoretical works suggested that SA could be important at shock waves as well (Ostrowski & Schlickeiser 1993; Schlickeiser & Achat 1993; Liu et al. 2008; Petrosian 2012; Afanasiev et al. 2014; Pohl et al. 2015). In this regard, Petrosian (2012) pointed out that some turbulence is usually produced at the shock front, and therefore SA by turbulence invariably plays a role in particle energization at the shock. According to some theoretical and numerical investigations, stochastic reacceleration behind the shock can account for the marked deviation of energy spectra of ions and electrons from the power law produced by the standard DSA (Afanasiev et al. 2014; Pohl et al. 2015). Nevertheless, these suggestions have not been tested against observations.

The heliosphere is the natural environment where acceleration theories can be tested against in situ observations. In particular, transients and corotating shocks are systems where particles are assumed to be accelerated as energetic particle enhancements are frequently associated with their passage (e.g., Armstrong et al. 1970; Sarris et al. 1976; Gosling et al. 1981; Pesses et al. 1982; Lario et al. 2003, and references therein). In the inner heliosphere, shock enhancements of protons are observed over a wide energy range from few tens of keV to some tens of MeV or even 100 MeV (e.g., Kallenrode 1996, and references therein). Nevertheless, the acceleration efficiency, i.e., the occurrence frequency of an enhancement at a particular energy, is a decreasing function of the energy (Tsurutani & Lin 1985; Lario et al. 2003; Dresing et al. 2016). The energetic particle intensities tend to increase with both the shock speed and the shock compression ratio. However, the hallmarks of the particle enhancements are not unequivocally determined by the shock parameters (Kallenrode 1996; Lario et al. 2005).

The particle enhancements at shock present a wide variety of different types according to their duration and the features of their time profiles (e.g., Lario et al. 2003). An interesting class of events consists of the so-called shock-spike events (SSEs), which are short-lived particle flux spikes observed at the passage of some interplanetary shocks in the inner heliosphere (e.g., Lanzerotti 1969; Sarris et al. 1976; Pesses et al. 1982). They typically last few tens of minutes and peak near the shock. SSEs may exhibit intensity enhancements up to energies of several tens of MeV. Though the first observations of SSEs date back almost half a century (e.g., Lanzerotti 1969), these kinds of events still deserve investigation, as particle acceleration in their high-energy range is not yet completely understood (e.g., Kallenrode 1995, 1996).

Here we investigate an SSE observed near Earth's orbit, on 2011 October 3 at 22: 23 UT, by the *STEREO B* spacecraft on the occasion of an interplanetary quasi-perpendicular shock passage. In Section 2 we show that the Weibull distribution represents a good description of the measured spectrum of the energetic particles, and in Section 3, we provide a theoretical derivation of the Weibull spectrum from a leaky-box model based on stochastic processes exhibiting anomalous diffusion for velocity (namely, processes in which particle velocity increases in time as a power law). In Section 4, we discuss how the experimental values of the spectrum parameters for the

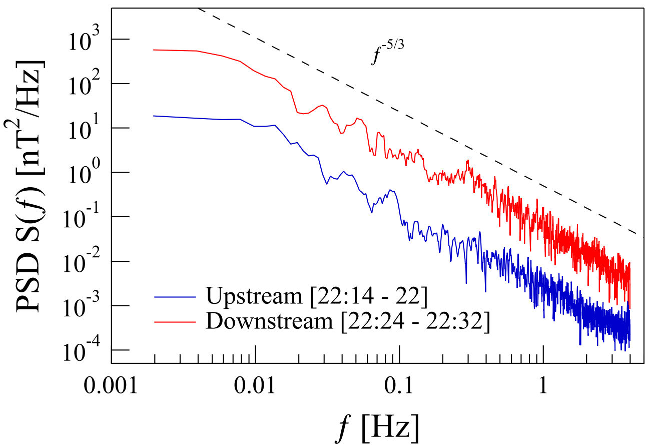


**Figure 2.** PSDs of the magnetic field fluctuations in the upstream and downstream regions of the shock observed on 2011 October 3 at 22:23:20 UT. The PSDs were calculated over the time intervals indicated. The dashed straight line $f^{-5/3}$, corresponding to a Kolmogorov turbulent spectrum, is drawn for reference.

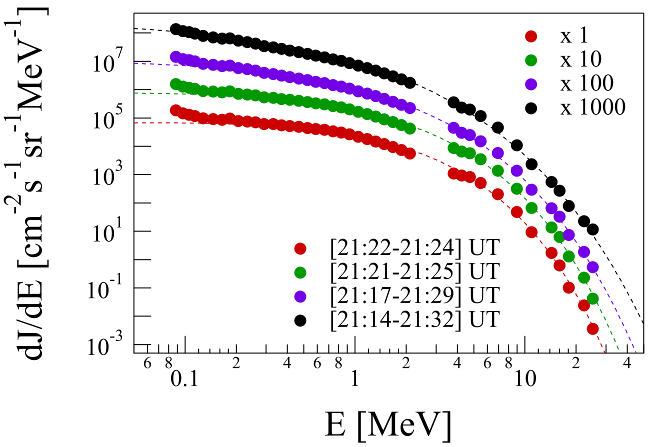


**Figure 3.** Time-averaged energetic particle differential fluxes calculated around the shock arrival on 2011 October 3 at 22:23:20 UT. The intervals over which the time averages were performed are indicated in the figure. Dashed curves are the best-fit Weibull functions. Data errors are within the marker size.

present SSE are coherent with the physical meaning they assume within the proposed scenario. Finally, we show that an alternative explanation of the observed Weibull spectrum can be provided in terms of SSA.

## 2. *STEREO B* Observations

On 2011 October 3 at 22:23:20 UT, the *STEREO B* spacecraft (located at 1.08 au, $-98\overset{\circ}{.}09$ and $1\overset{\circ}{.}08$ heliographic longitude and latitude, respectively) observed a quasi-perpendicular fast shock, with compression ratio and upstream Alfvénic Mach number $r_{sh} \simeq 2.3$ and $M_a \simeq 11$, respectively, moving radially outward from Sun at the speed of $v_{sh} \simeq 700$ km s$^{-1}$ with respect to the spacecraft reference frame.

At the same time, a particle enhancement was recorded by the Solar Electron Proton Telescope (SEPT; Müller-Mellin et al. 2008), the Low Energy Telescope (LET; Mewaldt et al. 2008) and the High Energy Telescope (HET; von Rosenvinge et al. 2008) in the energy range 0.1–30 MeV. Figure 1 reports a quick look of the main plasma and particle parameters along with magnetic field intensity measurements. The shock can be identified by the abrupt changes in the solar wind parameters. The upstream and downstream shock regions are both characterized by irregular magnetic fluctuations exhibiting power spectral densities (PSDs) with a power-law behavior $P(f) = f^{-q}$ (roughly with a Kolmogorov slope of $q \cong 5/3$) over three decades in frequency $f$ (0.001 $\div$ 0.1 Hz) (Figure 2). This behavior is commonly interpreted as the signature of the inertial range of an MHD turbulent cascade in the solar wind (e.g., Tu & Marsch 1995; Bruno & Carbone 2013). The post-shock power is an order of magnitude higher than the pre-shock one, indicating the enhancement of the fluctuation amplitude as a result of the shock passage.

Data used to study this event are 1-minute averaged proton fluxes measured by the three instruments in 39 energy differential channels. This event is characterized by spiky enhancements (up to energies of ∼30 MeV) with a time width of the order of tens of minutes occurring on a quite low intensity background (i.e., a background not associated with solar flare particles) and by a proton peak found near the shock passage at 22:23:20 UT (Figure 1). Such events associated with quasi-perpendicular shocks, called SSEs (e.g., Sarris et al. 1976; Pesses et al. 1982), are well-known phenomena in the vicinity of Earth's orbit, and their first observations date back to the end of the 1960s (e.g., Lanzerotti 1969). We underline that it is interesting to study SSEs because the acceleration in the MeV range and over is not yet completely understood in the interplanetary space.

An average differential flux $dJ/dE$ was calculated on the time interval 22:14–22:32 UT (around the shock arrival) covering the whole duration of the particle enhancement. Due to the large deviation of the $dJ/dE$ profile from a simple power law (Figure 3), a best fit was performed by means of a function derived from a Weibull spectrum (Laurenza et al. 2013, 2015, 2016):

$$N(E) = A(E/E_\tau)^{(\beta-1)} e^{-(E/E_\tau)^\beta} \quad (1)$$

($A$, $E_\tau$, and $\beta$ are constants), taking into account the conversion from the particle spectrum to the differential flux, namely, $dJ/dE = C \times N(E) \times E^{1/2}$. The resulting best-fit values for parameters were $C \sim [3.05 \pm 0.35] * 10^5$ cm$^{-2}$ s$^{-1}$ sr$^{-1}$ MeV$^{-1}$, $\beta = [0.50 \pm 0.01]$, and $E_\tau = [81 \pm 9]$ keV. The corresponding chi-square goodness of fit is $\chi^2 = 0.04$. Hence, as also shown in Figure 3, an excellent agreement exists between the Weibull spectrum and the experimental data over the wide energy range 0.1 $\div$ 30 MeV (spanning around two orders of magnitude) on which the best-fit procedure was performed. We point out that the Weibull spectrum continues to successfully fit ($\chi^2 < 0.1$) the particle fluxes even when the average spectrum is calculated over time intervals (as reported in Figure 3) smaller than the one considered above.

As the Weibull spectrum is not expected within mostly accepted acceleration theories in their standard formulation, it is worthwhile to investigate the physical context in which such a spectrum may arise. In the next section, we illustrate a theoretical connection of the Weibull spectrum with acceleration processes wherein the particle energy increases in time as a power law.

## 3. Weibull Spectrum: A Theoretical Derivation

The Weibull distribution (Weibull 1951) is a lifetime distribution that has application in many fields, such as reliability engineering, clinic trials, human dynamics, and economics (e.g., Rinne 2009; Fenner et al. 2015; Yannaros 1994, and references therein). The following derivation of the Weibull spectrum of energetic particles is essentially based on the connection between the Weibull distribution and "killed" processes exhibiting a power-law growth. Let us consider, for example, the simple case of the deterministic process $X(t) = t^{\nu}$ killed (viz., truncated) at a random time $T$ exponentially distributed according to the probability function $f(T) = e^{-T}$ . The killed state $\bar{X} = T^{\nu}$ has then a Weibull probability density $g(\bar{X}) = \beta\bar{X}^{\beta-1}e^{-\bar{X}^{\beta}}$ $(\beta = 1/\nu)$ as straight result of equating the probabilities $g(\bar{X})d\bar{X} = f(T)dT$. In the case of stochastic processes with power-law growth in expectation, the same basic idea can be applied. Nevertheless, the Weibull statistics is recovered only if the self-similarity of such processes is broken, as discussed below.

On the basis of the above argument, Fermi's SA can be directly related to the Weibull energy spectra since, in such physical process, the particle velocity increases in time as a power law. In fact, it is known that, in the framework of the second-order Fermi mechanism of acceleration, anomalous diffusion for particle velocity can arise (Bouchet et al. 2004; Perri et al. 2007): $\langle|\boldsymbol{v}(t) - \boldsymbol{v}_0|^2\rangle \sim t^{2\nu}$, where $\boldsymbol{v}_0$ is the initial velocity (the terminology "anomalous diffusion" refers only to the fact that the scaling exponents can differ from the standard Brownian value $\nu = 1/2$ although the random processes are Gaussian). For instance, Bouchet et al. (2004) developed a two-dimensional minimal stochastic model in which particles absorb kinetic energy (accelerate) through collisions against magnetic irregularities modeled as rigid, circular, randomly moving scattering centers. They found for both particle velocity and position the same anomalous superdiffusive behavior with $\nu = 1$. However, those authors noted that, in other systems with Fermi's acceleration, the scaling exponents for position and velocity diffusion could not be trivially related as in their minimal model. Values $1/2 < \nu \leqslant 1$ are possible in billiards with oscillating scatterers when noticeable displacements of scatterers are allowed and the collision rate depends on the particle(s) and scatterers' velocity (Kargovsky et al. 2013). More generally, Kargovsky et al. (2013) have shown that in the Langevin equation with multiplicative noise

$$\dot{v}(t) = av^{2\gamma-1} + v^{\gamma}(t)\xi(t) \quad (2)$$

(where $\gamma < 1$, $a \geqslant 0$, and $\xi(t)$ is a Gaussian white noise with $\overline{\xi(t)} = 0$ and $\overline{\xi(t)\xi(t+\tau)} = 2D\delta(\tau)$) the statistical moments of velocity grow in time according to the power law $M_n(t) \sim t^{n\nu}$ with $\nu = 1/(2 - 2\gamma)$. Those authors have pointed out that Equation (2) can describe, in the general case, a wide class of different physical systems in which the acceleration is due to both deterministic (first term on right-hand side) and random sources (second term on right-hand side).

In order to derive the Weibull spectrum, we assume that a set of noninteracting particles is accelerated by one of the stochastic processes described by Equation (2). Here the respective anomalous diffusion for velocity is expressed as

$$\langle E(t)^n\rangle \sim (t/\tau)^{n\nu(n)}, \quad (3)$$

where $E(t)$ is the particle kinetic energy, $n\nu(n)$ is a concave function of $n$ (i.e., its slope continually decreases), $\tau$ is a characteristic time of duration of the acceleration, and $\langle\bullet\rangle$ stands for the average over a particle's ensemble. The nonlinearity of $n\nu(n)$ indicates that the probability distribution function (PDF) of particle velocity at different times is not self-similar, namely, a PDF of the form $P(|\boldsymbol{v}|, t) = t^{-\nu}F(|\boldsymbol{v}|/t^{\nu})$ cannot describe the anomalous diffusion at all timescales by means of the same value of $\nu$. Actually, numerical studies on the motion of tracer particles in sandpile (Carreras et al. 1999) and in plasma turbulence (Carreras et al. 2001) show that system finite size effects can determine a breakdown of PDF self-similarity characterized by a nearly piecewise linear $n\nu(n)$ function with a smaller slope for high $n$ than for low $n$. Therefore, we justify the assumption of concavity for $n\nu(n)$ as a way to account for finite size effects on the velocity diffusion process (e.g., the finite value of the probability per unit time $\tau^{-1}$ for a particle to exit from the acceleration process).

Let us start our derivation from the classical Fermi scheme in which particles are stochastically accelerated in a spatial region by interactions with randomly moving scatterers representing magnetic irregularities or turbulent fluctuations. Moreover, let us assume that scattering is effective in making the particles distribution isotropic. The spatial region is homogeneous, and consequently the spatial diffusion is not considered. The number of particles per unit volume and per unit solid angle having kinetic energies in the range $E$ to $E + \Delta E$ is then expressed as $(4\pi)^{-1}N(E, t)\Delta E$, that is, only as a function of the time and energy. All of the particles are injected in the acceleration process with the same energy $E_{\rm in}$ (henceforth we refer all energies to $E_{\rm in}$ for notation convenience, thus $E_{\rm in} = 0$) at a constant rate of $q_{\rm in}$ particles per unit volume and time. Particle leakage from the acceleration region is taken into account through a characteristic time of confinement $\tau$ independent of the energy. Hence, the appropriate diffusion-loss equation, expressing the conservation of the number of particles in energy space, reads (e.g., Ginzburg & Syrovatskii 1964; Miller et al. 1990; Longair 1994)

$$\frac{\partial N}{\partial t} = \frac{\partial(b(E)N)}{\partial E} + \frac{1}{2}\frac{\partial^2(d(E)N)}{\partial E^2} - \frac{N}{\tau} + q_{\rm in}\delta(E), \quad (4)$$

where $b(E) = -\frac{d\langle E\rangle}{dt}$, $d(E) = \frac{d\langle(\Delta E)^2\rangle}{dt}$.

The four terms on the right-hand side account (left to right) for (1) the mean "drift" of the particles in energy space ($b(E)$ represents the average acceleration rate), (2) the "broadening" of the particle energy distribution (first and second right-hand-side terms are connected with the stochastic nature of the acceleration process), (3) particle leakage from the acceleration region (term accounting for the truncation in time or killing of the acceleration process), and (4) supply from sources of a monoenergetic beam of fresh particles with energy $E_{\rm in}$.

The relative weight of the second to the first term on the right-hand side of Equation (4) can be easily estimated, through dimensional considerations, by the ratio

$$R(\langle E\rangle) = \frac{d(\langle E\rangle)}{b(\langle E\rangle)\langle E\rangle} \sim \frac{\langle E^2\rangle}{\langle E\rangle^2}, \quad (5)$$

where we consider $\langle(\Delta E)^2\rangle \sim \langle E^2\rangle$. Hence, using Equation (3) in Equation (5) and dropping the bracket notation (hereafter no longer necessary), we obtain the scaling law:

$$R(S_F E) = S_F^{-2\alpha}R(E), \quad (6)$$

where $\alpha = [1 - \nu(2)/\nu(1)]$ and $S_F > 0$ is a scale factor.

Since $\alpha > 0$ due to the concavity of $n\nu(n)$, Equation (6) implies that $R(E) \ll 1$ if $E = S_F E_* \gg E_*$, where

$E_\tau = \langle E(\tau)\rangle$ and $\beta \equiv 1/\nu(1) > 0$, with $E_*$ being approximately defined through $R(E_*) \simeq 1$. Therefore, in energy regime $E \gg E_*$, the second term on the right-hand side of Equation (4) can be neglected and the steady-state spectrum ($\partial N/\partial t \equiv 0$) can be obtained by solving Equation (4) reduced to a simpler form:

$$N = -\frac{E_\tau^\beta}{\beta}\frac{\partial(E^{1-\beta}N)}{\partial E}, \tag{7}$$

where $E_\tau = \langle E(\tau)\rangle$ and $\beta \equiv 1/\nu(1) > 0$. A straightforward integration yields

$$N(E) = A(E/E_\tau)^{(\beta-1)}e^{-(E/E_\tau)^\beta}, \tag{8}$$

where $A$ is an integration constant. Therefore, the accelerated particles are distributed according to the Weibull statistics for energies sufficiently higher than $E_*$. On the contrary, in the low energy range from $E_{\rm in}$ to $\sim E_*$, the Weibull function is not expected to be a good approximation to the solution of the diffusion-loss equation, as the second term on the right-hand side of Equation (4) can no longer be neglected.

The stationary solution to Equation (4) is the equilibrium spectrum corresponding to the physical condition in which there exists a balance among particle escape, acceleration, and continuous injection from the source $q_{\rm in}\delta(E)$ switched on at an initial time $t_0$ in the past. We specify that in the case of shock acceleration, we consider our leaky box stuck on the shock front and moving with it and $t_0$, the time of shock formation. Such equilibrium is asymptotically reached when the elapsed time $t - t_0$ greatly exceeds the acceleration time for the maximum observed energy.

We note that $d(E) \equiv 0$ for a pure deterministic acceleration process. In this case, if the systematic time increase of the particle energy is a power law $E(t) \sim (t/\tau)^{1/\beta}$ (as in the case of SSA; Ohsawa 1987), Equation (4) automatically becomes Equation (7) when a stationary state is considered, and therefore the Weibull spectrum represents an exact equilibrium solution to the diffusion-loss equation.

As we have just shown, SA in a homogeneous leaky box with constant escape time determines a Weibull spectrum. In what follows we investigate the more general leaky box in which the spatial diffusion is taken into account, and we also illustrate the conditions under which the resulting corrections to the Weibull distribution of Equation (8) are negligible.

Spatial diffusion is introduced in the leaky box by adding the term $\kappa(E)\nabla^2 N(\boldsymbol{r}, E)$ on the right-hand side of Equation (4), where $\kappa(E)$ is the space diffusion coefficient. In turbulent plasmas, where the magnetic field fluctuations have a power density spectrum $W \propto k^{-q}$ ($k$ and $q$ are the wavenumber and the spectral index, respectively), the theoretical $\kappa(E)$ is a power law in energy (e.g., Miller et al. 1990; Fedorov et al. 2012), which can be written as $\kappa = \kappa_0(E/E_\tau)^{\gamma(q)}$ ($\kappa_0$ is a constant with dimensions $[l^2][t^{-1}]$). The function $\gamma(q)$ depends on the particular scattering model considered (e.g., Miller & Ramaty 1989; Droege 1994; O'Sullivan et al. 2009). The escape time $\tau_E$ due to the space diffusion $\langle \boldsymbol{r}^2(t)\rangle = \kappa\tau(t/\tau)^\lambda$ possesses a power-law dependence as well, as it holds the relation $L_{\rm AR}^2 = \kappa\tau(\tau_E/\tau)^\lambda$ between the typical size $L_{\rm AR}$ of the acceleration region and $\tau_E$, with $\tau$ being an arbitrary constant. The constant $\lambda$ may differ from the classical value $\lambda_{bm} = 1$ of the standard Brownian motion. In systems with stochastic particle acceleration such as the present one, the scaling exponent of the spatial diffusion $\lambda$ could be not trivially related to the scaling exponent of the diffusion in momentum space (here $\beta^{-1}$). The only general constraint is $\lambda \leqslant \beta^{-1} + 2$, and the equality holds only in the presence of a very strong correlation between the velocities at different times (Bouchet et al. 2004).

If we choose $\tau = L_{\rm AR}^2/\kappa_0$, the escape time can be expressed as $\tau_E = \tau(E/E_\tau)^{-\delta}$, where $\delta = \gamma/\lambda$. In this case, the equilibrium spectrum is the solution of an equation analogous to Equation (7):

$$\frac{N}{\tau_E} = -\frac{E_\tau^\beta}{\beta\tau}\frac{\partial(E^{1-\beta}N)}{\partial E} + \kappa(E)\nabla^2 N. \tag{9}$$

Equation (9) is solved through the method of the separation of variables. The straightforward result is the following:

$$\begin{aligned} N(\boldsymbol{r}, E) &= N_1(\boldsymbol{r})N_2(E) \\ &= N_1(\boldsymbol{r})A(E/E_\tau)^{(\beta-1)}e^{-G(\beta,\gamma,\lambda,E/E_\tau)(E/E_\tau)^{(\beta+\delta)}}, \end{aligned} \tag{10}$$

where $A$ is an integration constant, $N_1(\boldsymbol{r})$ a solution of equation $(-1/N_1)\kappa_0\nabla^2 N_1 = c = \text{const}$ ($c^{-1}$ is a typical diffusion time), and $G(\beta, \gamma, \lambda, E/E_\tau) = \beta/(\beta+\delta)[1 + c\tau(\beta+\delta)/(\beta+\gamma)(E/E_\tau)^{(\gamma-\delta)}]$.

In the case of particle spatial diffusion not much different from the Brownian motion, i.e., $\lambda \sim \lambda_{bm}$, we have $\delta \simeq \gamma$. Hence,

$$G \simeq \beta/(\beta+\delta)(1 + c\tau). \tag{11}$$

From Equations (11) and (10), provided that $\delta \ll \beta$, an approximate Weibull spectrum, with parameters $\beta$ and $E_\tau{}' = E_\tau/(1 + c\tau)^{1/\beta}$, is found even when the spatial diffusion by turbulence is accounted for in our leaky box.

On the other hand, for anomalous spatial diffusion, i.e., $\lambda \neq \lambda_{bm}$, if the condition $\delta \ll \beta$ is satisfied, a Weibull spectrum is still recovered whenever the escape time is much smaller than the typical diffusion time, that is, when $c\tau \ll 1$ holds.

Outside of the acceleration region, the particles are only scattered by the magnetic field inhomogeneities, and their energy is unchanged. The equilibrium spectrum is obtained by solving the diffusion-loss equation, which, in this case, describes a pure spatial diffusion out of the source:

$$\kappa_{\rm out}(E)\nabla^2 N = 0, \tag{12}$$

where $\kappa_{\rm out}(E)$ is the spatial diffusion coefficient outside the acceleration region.

The analytical form of the solution $N_{\rm out}(\boldsymbol{r}, E)$ of this equation depends on the boundary conditions related to the geometry of the particular problem studied. However, it is possible to highlight a qualitative and general property of this solution independent from the details of the specific physical picture. In fact, at any space point $\boldsymbol{r}$, the particle spectrum $N_{\rm out}(\boldsymbol{r}, E)$ has to be such as to ensure the conservation of the number of particles of energy $E$ diffusing out from the spatial source where they are produced. This constraint implies that $N_{\rm out}(\boldsymbol{r}, E)$ is a decreasing function of the distance $r$ from the acceleration region.

## 4. Discussion

Theoretical considerations, numerical simulations, and observations (e.g., McKenzie & Westphal 1968; Westphal & McKenzie 1969; Kennel et al. 1982; Zank et al. 2006;

Giacalone & Jokipii 2007; Lu et al. 2009; Guo et al. 2012; Hu et al. 2012, 2013; Fraschetti 2013; Pohl et al. 2015) indicate that shock waves produce substantial levels of turbulent fluctuations and structures in their downstream regions. Therefore, the two fundamental assumptions, on which our derivation of the Weibull spectrum is based, are consistent with physical conditions at interplanetary shocks. In fact, from the theoretical point of view, turbulence can provide efficient particle scattering (thus supporting our first basic assumption; e.g., Tverskoĭ 1968; Blandford & Eichler 1987; Petrosian 2012; Bykov et al. 2014) to account for the observed isotropy of the distribution function of energetic particles (Gosling et al. 1981) and the particle confinement in a volume close to the shock front. In addition, turbulence can be responsible for momentum diffusion (our second basic assumption; Tverskoĭ 1968; Bouchet et al. 2004), so that the energy is transferred to particles through a stochastic Fermi acceleration (e.g., Tverskoĭ 1968; Fedorov et al. 2012; Petrosian 2012) achieved through adiabatic particle reflection from randomly moving turbulent waves or eddies.

In the previous section, we have shown that in turbulent plasmas the space diffusion leads to a particle spectrum that could be approximated by Weibull's functional form. As a matter of fact, our observations indicate that all of the average spectra calculated on different time intervals around the shock are well described by a Weibull function in the energy range $[0.1 \div 30]$ MeV (Figure 3). In addition, we verified that the goodness of our fits is not improved when the spectrum of Equation (10) is chosen as the fitting function. Hence, presumably the condition $\delta \ll \beta$, which ensures negligible corrections to the Weibull distribution, is satisfied. In particular, as $\beta$ is found to be ~0.5 for all of the analyzed spectra, $\delta$ is possibly close to zero. Therefore, in spite of extreme simplicity, an energy-independent escape time (i.e., $\delta = 0$) proves to be (a posteriori) acceptable as a result of the good agreement between the Weibull spectrum and experimental data. In this regard, we also note that, in the past years, a constant leaky-box lifetime has been extensively and fruitfully applied in studies of the emission of solar flare particles accelerated through the Fermi stochastic mechanism (Miller et al. 1990, and references therein).

In our theoretical scheme $\tau$, $\beta$, and $E_\tau$ are free parameters that can assume, in principle, any value independently from each other. We show that their observational estimates are congruent with a physical picture of the event by discussing the case of the average spectrum calculated on the interval 22:14–22:31 UT. However, the same conclusion is valid also for all of the average spectra calculated on shorter intervals around the peak. The value of $\beta = 0.5$ (i.e., $\nu(1) = 2$) implies superdiffusion for velocity. In general, a high degree of persistence of the anomalous diffusion is expected for an efficient particle acceleration. Moreover, as already mentioned, the same superdiffusive behavior spontaneously arises in a minimal model of second-order Fermi acceleration proposed by Bouchet et al. (2004). Thus, the above $\beta$ value proves to be fairly meaningful from a physical point of view.

In the case of efficient energization, the mean energy $E_\tau$ gained in a characteristic time $\tau$ has to be much higher than the typical injection energy. As a matter of fact, $E_\tau = 81$ keV considerably exceeds both typical bulk flow $E_{\text{bulk}} = 1/2 m_p V_{\text{sw}}^2 \sim 1$ keV and thermal $E_{\text{th}} = K_B T_p \sim 0.02$ keV energies of the upstream solar wind protons (see Figure 1). Hence, it is consistent with the reasonable hypothesis that the energetic particle population is accelerated directly out of the ambient solar wind.

The confinement time $\tau$ cannot be directly obtained through the best-fit procedure. Nevertheless, observations can provide an upper limit for its value. In fact, taking into account $\beta = 0.5$ and $E_\tau = 81$ keV, it is seen from Equation (3) that a particle energy of ~30 MeV (i.e., the highest energy in Figure 3) is reached after an acceleration time $T_a \simeq 18\tau$. The shock traveling time, from the Sun to the spacecraft position $R_{s/c} = 1.08$ au, is $T_{\text{stt}} = R_{s/c}/v_{\text{sh}} \simeq 2.7$ days. Hence, as the condition $T_{\text{stt}} \gg T_a$ must be satisfied in order to observe the Weibull equilibrium spectrum (see the discussion on this question in Section 3), a reasonable upper limit is (assuming the conservative constraint $T_{\text{stt}} \sim 10 T_a$) $\tau_{\text{up}} \simeq 0.4$ hr. When calculated from Equation (3) with the above values of $\beta$, $E_\tau$, and $\tau$, the resulting acceleration timescales of our superdiffusive model are comparable to DSA ones or even shorter. For instance, Zhang & Lee (2013) estimate that DSA accelerates a proton to an energy of ~10 MeV in a time of ~12 hr at 1 au (see their Figure 1). In our case, the same energy is reached after a time $\sim 10\tau$, i.e., ~4 hr. It is conceivable that a second-order Fermi acceleration may be as efficient as the DSA. For instance, Ostrowski (1994) has showed that, under the hypothesis of negligible damping of very low frequency Alfvén waves, statistical acceleration by high-amplitude MHD turbulence can transfer the energy of a weak parallel shock to the particles more efficiently than a first-order process. Moreover, Schlickeiser & Achat (1993) proposed that, due to efficient momentum diffusion of particles in the downstream region of the shock, the acceleration can be dominated by the second-order acceleration mechanism. As far as the time profile of the energetic particle flux is concerned, we note a different behavior between low- and high-energy channels. In particular, for energies up to about 0.5 MeV, the profile stays nearly constant in the downstream region after peaking at the shock, as expected for DSA (e.g., Zank et al. 2006). On the other hand, at higher energies the flux profile drops soon after the shock arrival. Such a behavior could be qualitatively explained through a steady spatial diffusion of energetic particles out of a narrow acceleration region behind the shock as briefly discussed in Section 3. Hence, as the space diffusion coefficient due to scattering is an increasing function of the energy (e.g., Fedorov et al. 2012), the post-shock flux would show a time decrease more pronounced going from the lowest to highest energies when observed at a fixed spacecraft position. In this regard, we point out that fast hydromagnetic waves could be consistently taken into account within this physical picture. In fact, strong turbulent forward- and backward-moving fast modes are expected to be excited downstream of the shock in a potentially thin layer of the order of the injection scale of the fast turbulence driven at the shock. As a result of their high phase speed, such fast modes can produce SA or reacceleration (if DSA is effective at an early stage) until they are damped away (Liu et al. 2008; Pohl et al. 2015, and references therein). The above considerations suggest that SA plays a role in particle acceleration and determines the form of the energetic particle spectrum, possibly together with contribution of DSA at low energies (of the order of $E_\tau$). Nevertheless, this speculation needs to be supported by further theoretical studies to address the question of the driving of the turbulence at the shock and how the microphysics of the turbulence affects the trapping and the acceleration time of energetic particles (e.g., Verkhoglyadova & Le Roux 2005; Liu et al. 2008; Pohl et al. 2015). The experimental identification (through high-resolution plasma data) of the type of the turbulent waves near the shock front would be fundamental to understanding the particle acceleration as well.

An alternative interpretation of the present observations is provided by SSA. In fact, according to the SSA at nearly perpendicular shocks, the particle energy augments as the power law $E(t) \sim t^2$ (Ohsawa 1987; Lee et al. 1996). Consequently, as explained in the previous section under the hypothesis of an acceleration time that is a random variable, the expected particle spectrum is a Weibull one with $\beta = 1/2$, which is exactly the same value obtained from the best-fit procedure for the average spectrum of Figure 3, corresponding to the time interval covering the whole particle enhancement. Moreover, taking into account the upstream values of the Alfvén speed $v_A \simeq 65$ km s$^{-1}$, Mach number $M_A \simeq 11$, and proton-to-electron mass ratio $m_p/m_e = 1836$, the estimate of the theoretical maximum proton speed (Ohsawa 1987) $v_{\max} \simeq v_A (m_p/m_e)^{1/2}(M_A - 1)^{3/2}$ is $v_{\max} \simeq 88 \times 10^3$ km s$^{-1}$, corresponding to a kinetic energy of $E_{\max} \simeq 38$ MeV, i.e., a value close to ~30 MeV of the higher energy observed in the present event. Also, $E_{\max}$ is reached in a time $t_{\max} \simeq (m_p/m_e)^{1/2}\omega_p{}^{-1}M_A{}^{-1}$ (Ohsawa 1987), which, in our case, is 125 s (with the upstream proton cyclotron angular frequency being $\omega_p = 0.95$ rad s$^{-1}$). Taking into account $t_{\max}$, $E_{\max}$, and $E_\tau$, an elementary calculation using Equation (3) yields a characteristic acceleration time $\tau \simeq 6$ s, which, from the physical viewpoint, is a reasonable value considering that the proton gyroperiod is $2\pi/\omega_p \simeq 7$ s. Therefore, SSA captures some important aspects of the study reported here. Nevertheless, no definitive conclusions in this regard can be drawn without further confirmations of the characteristic value $\beta \simeq 0.5$ from other events of particle acceleration at perpendicular shocks.

Because of the physical connection between the Weibull spectrum and stochastic or SSA, the present event seems to support the idea that in the MeV range and above, in which a Weibull distribution well describes our SSE observations, particle acceleration could be provided by different mechanism(s) from those at work from a few tens up to a few hundreds of keV (Kallenrode 1995, 1996). In fact, in the latter energy range, many observational evidences point to an interpretation of the shock enhancements in terms of DSA or pump acceleration (Fisk & Gloeckler 2012; Giacalone 2012; Neergaard Parker & Zank 2012).

## 5. Conclusions

With regard to the SSE of particle acceleration in the interplanetary space presented here, our results can be summarized as follows.

(1) A Weibull distribution successfully fits the spectrum of energetic protons over the entire observed energy range $0.1 \div 30$ MeV spanning two orders of magnitude.
(2) The Weibull spectrum can be theoretically derived from a leaky-box model in the framework of stochastic processes (including the classical Fermi second-order mechanism), wherein the acceleration is mathematically represented as an anomalous diffusion in momentum space. A well-known Langevin equation with Gaussian multiplicative white noise can describe a wide class of such processes.
(3) An overall coherence exists between the experimental values of the Weibull spectrum free parameters and their physical meaning within the proposed statistical mechanism. Moreover, we found that the timescales of the SA could be competitive with those of the DSA.
(4) All of the deterministic acceleration processes, characterized by a time increase of the particle energy according to a power law, lead to a Weibull spectrum in the case in which the acceleration time is assumed to be a random variable. In particular, the present SSE can be interpreted in terms of SSA as well. Our results show that such a description makes sense from the physical point of view. Nevertheless, before suggesting SSA as a viable mechanism for acceleration at quasi-perpendicular shocks connected with SSEs, we would need to ascertain whether the characteristic SSA parameter $\beta \simeq 0.5$ of the Weibull spectrum is routinely observed in space.
(5) The present results suggest a scenario in which different mechanisms could account for particle acceleration in different energy ranges at interplanetary shocks, namely, stochastic or SSA effective in the MeV range and above, whereas (as reported in past studies) diffusive, pump, or shock-drift acceleration is at work for energies from few tens up to few hundreds of keV.

The authors would like to thank all of the *STEREO* teams for making the data available and M. Storini for the useful discussions. This research has been supported by the European Community's Seventh Framework Programme ([FP7/2007-2013]) under Grant agreement no. 313038/STORM, ASI/INAF contract no. I/022/10/0, and the Italian MIUR-PRIN grant 2012P2HRCR on "The Active Sun and Its effects on Space and Earth Climate."

## References

Afanasiev, A., Vainio, R., & Kocharov, L. 2014, ApJ, 790, 36
Armstrong, T. P., Krimigis, S. M., & Behannon, K. W. 1970, JGR, 75, 5980
Blandford, R., & Eichler, D. 1987, PhR, 154, 1
Blandford, R. D., & Ostriker, J. P. 1978, ApJL, 221, L29
Bouchet, F., Cecconi, F., & Vulpiani, A. 2004, PhRvL, 92, 040601
Bruno, R., & Carbone, V. 2013, LRSP, 10, 2
Bykov, A. M., Ellison, D. C., Osipov, S. M., & Vladimirov, A. E. 2014, ApJ, 789, 137
Carreras, B. A., Lynch, V. E., Newman, D. E., & Zaslavsky, G. M. 1999, PhRvE, 60, 4770
Carreras, B. A., Lynch, V. E., & Zaslavsky, G. M. 2001, PhPl, 8, 5096
Decker, R. B., & Vlahos, L. 1985, JGR, 90, 47
Dresing, N., Theesen, S., Klassen, A., & Heber, B. 2016, A&A, 588, A17
Droege, W. 1994, ApJS, 90, 567
Eilek, J. A. 1979, ApJ, 230, 373
Ellison, D. C., & Ramaty, R. 1985, ApJ, 298, 400
Fedorov, Y., Shakhov, B., & Stehlik, M. 2012, JPhB, 45, 165702
Fenner, T., Levene, M., & Loizou, G. 2015, JSMTE, 8, 08015
Fermi, E. 1949, PhRv, 75, 1169
Fisk, L. A., & Gloeckler, G. 2012, SSRv, 173, 433
Fisk, L. A., & Gloeckler, G. 2014, JGRA, 119, 8733
Fraschetti, F. 2013, ApJ, 770, 84
Giacalone, J. 2012, ApJ, 761, 28
Giacalone, J., & Jokipii, J. R. 2007, ApJL, 663, L41
Ginzburg, V. L., & Syrovatskii, S. I. 1964, The Origin of Cosmic Rays (New York: Macmillan)
Gosling, J. T., Asbridge, J. R., Bame, S. J., et al. 1981, JGR, 86, 547
Guo, F., Li, S., Li, H., et al. 2012, ApJ, 747, 98

Hu, Q., Ao, X., Peltzer, R., & Zank, G. P. 2012, in AIP Conf. Ser. 1500, Space Weather: The Space Radiation Environment, ed. Q. Hu et al. (Melville, NY: AIP), 192
Hu, Q., Zank, G. P., Li, G., & Ao, X. 2013, in AIP Conf. Ser. 1539, Solar Wind 13, ed. G. P. Zank et al. (Melville, NY: AIP), 175
Kallenrode, M.-B. 1995, AdSpR, 15, 375
Kallenrode, M.-B. 1996, JGR, 101, 24393
Kargovsky, A. V., Anashkina, E. I., Chichigina, O. A., & Krasnova, A. K. 2013, PhRvE, 87, 042133
Kennel, C. F., Coroniti, F. V., Scarf, F. L., Smith, E. J., & Gurnett, D. A. 1982, JGR, 87, 17
Krymskii, G. F. 1977, DoSSR, 234, 1306
Lanzerotti, L. J. 1969, JGR, 74, 2851
Lario, D., Ho, G. C., Decker, R. B., et al. 2003, in AIP Conf. Ser. 679, Solar Wind Ten, ed. M. Velli et al. (Melville, NY: AIP), 640
Lario, D., Hu, Q., Ho, G. C., et al. 2005, in ESA Special Publication 592, Solar Wind 11/SOHO 16, Connecting Sun and Heliosphere, ed. B. Fleck, T. H. Zurbuchen, & H. Lacoste (Noordwijk: ESA Publications Division), 81
Laurenza, M., Consolini, G., Storini, M., & Damiani, A. 2013, in AIP Conf. Ser. 1539, Solar Wind 13, ed. G. P. Zank et al. (Melville, NY: AIP), 219
Laurenza, M., Consolini, G., Storini, M., & Damiani, A. 2015, JPhCS, 632, 012066
Laurenza, M., Consolini, G., Storini, M., Pallocchia, G., & Damiani, A. 2016, JPhCS, 767, 012015
Lee, M. A., Mewaldt, R. A., & Giacalone, J. 2012, SSRv, 173, 247
Lee, M. A., Shapiro, V. D., & Sagdeev, R. Z. 1996, JGR, 101, 4777
Lichtenberg, A. J., & Lieberman, M. A. 1991, Regular and Chaotic Dynamics (2nd ed.; New York, NY: Springer-Verlag)
Liu, S., Fan, Z.-H., Fryer, C. L., Wang, J.-M., & Li, H. 2008, ApJL, 683, L163
Longair, M. S. 1994, High Energy Astrophysics. Vol. 2: Stars, The Galaxy and the Interstellar Medium (2nd ed.; Cambridge: Cambridge Univ. Press)
Lu, Q., Hu, Q., & Zank, G. P. 2009, ApJ, 706, 687
McKenzie, J. F., & Westphal, K. O. 1968, PhFl, 11, 2350
Mewaldt, R. A., Cohen, C. M. S., Cook, W. R., et al. 2008, SSRv, 136, 285
Mewaldt, R. A., Cohen, C. M. S., Labrador, A. W., et al. 2005, JGRA, 110, A09S18
Michalek, G., Ostrowski, M., & Schlickeiser, R. 1999, SoPh, 184, 339
Miller, J. A., Guessoum, N., & Ramaty, R. 1990, ApJ, 361, 701
Miller, J. A., & Ramaty, R. 1989, ApJ, 344, 973
Müller-Mellin, R., Böttcher, S., Falenski, J., et al. 2008, SSRv, 136, 363
Neergaard Parker, L., & Zank, G. P. 2012, ApJ, 757, 97
Neergaard Parker, L., Zank, G. P., & Hu, Q. 2014, ApJ, 782, 52
O'Sullivan, S., Reville, B., & Taylor, A. M. 2009, MNRAS, 400, 248
Ohsawa, Y. 1986, PhFl, 29, 773
Ohsawa, Y. 1987, GeoRL, 14, 95
Ostrowski, M. 1994, A&A, 283, 344
Ostrowski, M., & Schlickeiser, R. 1993, A&A, 268, 812
Perri, S., Lepreti, F., Carbone, V., & Vulpiani, A. 2007, EPL, 78, 40003
Pesses, M. E., Decker, R. B., & Armstrong, T. P. 1982, SSRv, 32, 185
Petrosian, V. 2012, SSRv, 173, 535
Petrosian, V., & Liu, S. 2004, ApJ, 610, 550
Pohl, M., Wilhelm, A., & Telezhinsky, I. 2015, A&A, 574, A43
Rinne, H. 2009, The Weibull Distribution: A Handbook (Boca Raton, FL: CRC Press)
Sagdeev, R. Z. 1966, RvPP, 4, 23
Sarris, E. T., Krimigis, S. M., & Armstrong, T. P. 1976, GeoRL, 3, 133
Schlickeiser, R., & Achat, U. 1993, JPlPh, 50, 85
Scott, J. S., & Chevalier, R. A. 1975, ApJL, 197, L5
Seo, E. S., & Ptuskin, V. S. 1994, ApJ, 431, 705
Tsurutani, B. T., & Lin, R. P. 1985, JGR, 90, 1
Tu, C.-Y., & Marsch, E. 1995, SSRv, 73, 1
Tverskoĭ, B. A. 1968, JETP, 26, 821
van Nes, P., Reinhard, R., Sanderson, T. R., Wenzel, K.-P., & Zwickl, R. D. 1984, JGR, 89, 2122
Verkhoglyadova, O. P., & Le Roux, J. A. 2005, JGRA, 110, A10S03
von Rosenvinge, T. T., Reames, D. V., Baker, R., et al. 2008, SSRv, 136, 391
Weibull, W. 1951, JAM, 18, 293
Westphal, K. O., & McKenzie, J. F. 1969, PhFl, 12, 1228
Yannaros, N. 1994, Ann. Inst. Stat. Math., 46, 641
Zank, G. P., Hunana, P., Mostafavi, P., et al. 2015, ApJ, 814, 137
Zank, G. P., Li, G., Florinski, V., et al. 2006, JGRA, 111, 6108
Zaslavskii, G. M., & Chirikov, B. V. 1965, SPhD, 9, 989
Zhang, M., & Lee, M. A. 2013, SSRv, 176, 133