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# Energy-momentum tensor and helicity for gauge fields coupled to a pseudoscalar inflaton

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We study the energy-momentum tensor and helicity of gauge fields coupled through  $g\phi F\tilde{F}/4$  to a pseudoscalar field  $\phi$  driving inflation. Under the assumption of a constant time derivative of the background inflaton, we compute analytically divergent and finite terms of the energy density and helicity of gauge fields for any value of the coupling  $g$ . We introduce a suitable adiabatic expansion for mode functions of physical states of the gauge fields which correctly reproduces ultraviolet divergences in average quantities and identifies corresponding counterterms. Our calculations shed light on the accuracy and the range of validity of approximated analytic estimates of the energy density and helicity terms previously existed in the literature in the strongly coupled regime only, i.e., for  $g\dot{\phi}/(2H) \gg 1$ . We discuss the implications of our analytic calculations for the backreaction of quantum fluctuations onto the inflaton evolution.

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## I. INTRODUCTION

Inflation driven by a real single scalar field (inflaton) slowly rolling on a smooth self-interaction potential represents the minimal class of models in general relativity which are in agreement with observations. Not only are the details of the fundamental nature of the inflaton and of its interaction with other fields needed to study the stage of reheating after inflation, but also they can be important for theoretical and phenomenological aspects of its evolution.

Axion inflation, and more generally, inflation driven by a pseudoscalar field is the archetypal model to include parity violation during a nearly exponential expansion and it has a rich phenomenology. An interaction of the pseudoscalar field with gauge fields of the type

$$\mathcal{L}_{\text{int}} = -\frac{g\phi}{4} F^{\mu\nu} \tilde{F}_{\mu\nu}, \quad (1)$$

where  $g$  is a coupling constant with a physical dimension of length, or inverse energy (we put  $\hbar = c = 1$ ), leads to decay of the pseudoscalar field into gauge fields modifying its background dynamics [1] and to a wide range of potentially observable signatures including primordial magnetic fields [2–10], preheating at the end of inflation [4, 11, 12], baryogenesis and leptogenesis [13–15], equilateral non-Gaussianities [16–18], chiral gravitational waves in the range of direct detection by gravitational wave antennas [19–22], and primordial black holes [18, 23–26].

The decay of the inflaton into gauge fields due to the coupling in Eq. (1) is a standard problem of amplification of quantum fluctuation (gauge fields) in an external classical field (the inflaton). Applications of the textbook regularization techniques used for calculation of quantum effects in curved space-time [27, 28] have led to interesting novel results in the de Sitter [29] and inflationary space-times [30–32], also establishing a clear connection between the stochastic approach [33, 34] and field theory methods [35, 36].

In this paper, we apply the technique of adiabatic regularization [37, 38]<sup>1</sup> to the energy density and helicity of gauge fields generated through the interaction in Eq. (1).

<sup>1</sup>It was called “n-wave regularization” in [37].

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The evolution equation of gauge fields admits analytical solutions under the assumption of a constant  $\xi \equiv g\dot{\phi}/(2H)$  [5], where  $H \equiv \dot{a}/a$  is the Hubble parameter during inflation, also considered constant in time. Here we solve in an analytical way for the averaged energy density and helicity of gauge fields for any value of  $\xi$ . Previously, only approximate results valid in the strongly coupled regime  $\xi \gg 1$  were obtained in the literature. Our technique of computing integrals in the Fourier space is based on previous calculations of the Schwinger effect for a  $U(1)$  gauge field in the de Sitter space-time [39–42], but now it is applied to a novel problem in which the classical external field is the inflaton. More recent papers apply similar techniques to calculate backreaction of  $SU(2)$  gauge fields [43,44] and fermions [45,46] on the de Sitter space-time.

Our paper is organized as follows. In Secs. II and III, we review the basic equations and the averaged energy-momentum tensor and helicity of the gauge fields, respectively. In Sec. IV, we present analytical results for the bare averaged quantities, and we direct the interested reader to Appendixes A and B for more detailed calculations. In Sec. V, we outline the adiabatic regularization scheme used (see also [32]). We also show that counterterms appearing in the adiabatic subtraction method can be naturally interpreted as coming from renormalization of self-interaction terms of the scalar field either existing in the bare Lagrangian density, or those which have to be added to it due to the nonrenormalizability of the problem involved (that is clear from  $g$  being dimensional). We then describe the implications of our results to the homogeneous dynamics of inflation with the backreaction of one-loop quantum effects taken into account in Sec. VI, particularly focusing on the new regime of validity  $|\xi| \lesssim 1$  and commenting on the differences from previous results existing in the literature. We then conclude in Sec. VII.

## II. SETTING OF THE PROBLEM

The Lagrangian density describing a pseudoscalar inflaton field  $\phi$  coupled to a  $U(1)$  gauge field is

$$\mathcal{L} = -\frac{1}{2}(\nabla\phi)^2 - V(\phi) - \frac{1}{4}(F^{\mu\nu})^2 - \frac{g\phi}{4}F^{\mu\nu}\tilde{F}_{\mu\nu}, \quad (2)$$

where  $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2 = \epsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)/2$  and  $\nabla$  is the metric covariant derivative. We consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric  $ds^2 = -dt^2 + a^2 d\mathbf{x}^2$ , where  $a(t)$  is the scale factor, and we write the coupling constant  $g = \alpha/f$ , where  $f$  is the axion decay constant. We consider gauge fields to linear order in a background driven only by a nonzero time-dependent *vev*  $\phi(t)$ .

In this context, it is convenient to adopt the basis of circular polarization  $\epsilon_\pm$  transverse to the direction of propagation defined by the comoving momentum  $\mathbf{k}$ . In the Fourier space, we then have

$$\mathbf{k} \cdot \epsilon_\pm = 0, \quad (3)$$

$$\mathbf{k} \times \epsilon_\pm = \mp i|k|\epsilon_\pm. \quad (4)$$

Expanding the second quantized gauge field in terms of creation and annihilation operators for each Fourier mode  $\mathbf{k}$ , we get

$$\mathbf{A}(t, \mathbf{x}) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} [\epsilon_\lambda(\mathbf{k})A_\lambda(\tau, \mathbf{k})a_\lambda^{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + \text{H.c.}], \quad (5)$$

where the Fourier mode functions  $A_\pm$  for the two circular polarizations satisfy the following equation of motion:

$$\frac{d^2}{d\tau^2}A_\pm(\tau, k) + (k^2 \mp gk\phi')A_\pm(\tau, k) = 0. \quad (6)$$

Here the prime denotes the derivative with respect to the conformal time  $\tau$  ( $d\tau = dt/a$ ).

The above equation admits a simple analytic solution for a constant  $\dot{\phi}$  ( $\equiv d\phi/dt$ ) in a nearly de Sitter stage during inflation. A constant time derivative for the inflaton evolution can be obtained for  $V(\phi) = \Lambda^4(1 - C|\phi|)$  with  $|C\phi| \ll 1$ , or for  $V(\phi) \propto m^2\phi^2$  [47,48]. Natural inflation with  $V(\phi) = \Lambda^4[1 \mp \cos(\phi/f)]$  can be approximated better and better by  $m^2\phi^2$  for  $f \gg M_{pl}$  with  $m = \Lambda^2/f$ , which is the regime allowed by cosmic microwave background (CMB henceforth) anisotropy measurements [49,50].

We thus study the inflationary solution assuming a de Sitter expansion, i.e.,  $a(\tau) = -1/(H\tau)$ , with  $\tau < 0$ ,  $H \simeq \text{const}$  and  $\dot{\phi} \simeq \text{const}$ . In such a case, we can write  $\phi' \simeq -\sqrt{2\epsilon_\phi}M_{pl}/\tau$  with  $\epsilon_\phi = \dot{\phi}^2/(2M_{pl}^2H^2)$  being one of the slow-roll parameters. In this case, the equation of motion for the two circular polarization mode functions becomes [1]

$$\frac{d^2}{d\tau^2}A_\pm(\tau, k) + \left(k^2 \pm \frac{2k\xi}{\tau}\right)A_\pm(\tau, k) = 0, \quad (7)$$

where  $\xi \equiv g\dot{\phi}/(2H)$ . The above equation reduces to ([51] p. 538) with  $L = 0$  and admits a solution in terms of the regular and irregular Coulomb wave functions corresponding to the positive frequency for  $-k\tau > 0$ ,

$$A_\pm(\tau, k) = \frac{[G_0(\pm\xi, -k\tau) + iF_0(\pm\xi, -k\tau)]}{\sqrt{2k}}. \quad (8)$$

These can be rewritten in the subdomain  $-k\tau > 0$  in terms of the Whittaker  $W$  functions,

$$A_\pm(\tau, k) = \frac{1}{\sqrt{2k}}e^{\pm\pi\xi/2}W_{\pm i\xi, \frac{1}{2}}(-2i k\tau). \quad (9)$$

Note that the above solutions are symmetric under the change  $A_+ \rightarrow A_-$  and  $\xi \rightarrow -\xi$  in Eq. (7).

### III. ENERGY-MOMENTUM TENSOR AND HELICITY

From the Lagrangian density in Eq. (2), it is easy to derive the metric energy-momentum tensor (EMT) for the gauge-fields

$$T_{\mu\nu}^{(F)} = F_{\rho\mu}F^{\rho\nu} + g_{\mu\nu} \frac{\mathbf{E}^2 - \mathbf{B}^2}{2}, \quad (10)$$

with  $\mathbf{E}$  and  $\mathbf{B}$  being the associated electric and magnetic field. We then obtain for the energy density and pressure the following expressions:

$$T_{00}^{(F)} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2}, \quad (11)$$

$$T_{ij}^{(F)} = -E_i E_j - B_i B_j + \delta_{ij} \frac{\mathbf{E}^2 + \mathbf{B}^2}{2}. \quad (12)$$

Note that there are no terms in the EMT depending on the pseudoscalar coupling. Their absence in the  $T_{00}^{(F)}$  component follows from impossibility to construct a pseudoscalar invariant under spatial rotations from  $\mathbf{E}$  and  $\mathbf{B}$ . Then the fact that the EMT trace remains zero in the presence of the interaction of Eq. (1), due to the conformal invariance of the gauge field, leads to the absence of such terms in  $T_{ij}^{(F)}$ , too.

By using Eqs. (11) and (12), the Friedmann equations take the form

$$H^2 = \frac{1}{3M_{pl}^2} \left[ \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} \right], \quad (13)$$

$$\dot{H} = -\frac{1}{2M_{pl}^2} \left[ \dot{\phi}^2 + \frac{2}{3} \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle \right]. \quad (14)$$

Using the relations  $\vec{E} = -\vec{A}'/a^2$  and  $\vec{B} = \vec{\nabla} \times \vec{A}'/a^2$ , the averaged energy density is

$$\begin{aligned} \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} &= \int \frac{dk}{(2\pi)^2 a^4} I(k) \\ &= \int \frac{dk}{(2\pi)^2 a^4} k^2 [ |A'_+|^2 + |A'_-|^2 \\ &\quad + k^2 (|A_+|^2 + |A_-|^2) ]. \end{aligned} \quad (15)$$

The electric (magnetic) contribution is given by the first and second (third and fourth) terms in the integrand. It is easy to see that this integral diverges for large momentum  $k$ . This is a common behavior for averaged quantities in quantum field theory (QFT henceforth) in curved background, or in external fields, and a renormalization procedure is needed to remove these UV divergences. In Sec. V, we will use the adiabatic regularization method [37,38] for this purpose,

and we will present counterterms needed to renormalize the bare constants in the Lagrangian in Sec. VA. In the present section, we identify the UV divergent contributions in the integrands.

Expanding the integrand of Eq. (15) for  $-k\tau \gg 1$ , we obtain quartic, quadratic, and logarithmic UV divergences,

$$I_{\text{div}}(k) \sim 2k^3 + \frac{\xi^2 k}{\tau^2} + \frac{3\xi^2(-1 + 5\xi^2)}{4\tau^4 k} + \mathcal{O}\left(\frac{1}{k}\right)^3. \quad (16)$$

It is interesting to note that the logarithmic divergence changes its sign when  $\xi$  crosses  $|\xi| = 1/\sqrt{5}$ . On the other hand, expanding in the IR limit ( $-k\tau \ll 1$ ), the integrand of Eq. (15) has no IR divergences.

The equation of motion for the inflation  $\phi$  is affected by the backreaction of these gauge fields,

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = g\langle \mathbf{E} \cdot \mathbf{B} \rangle, \quad (17)$$

where the helicity integral is given by

$$\begin{aligned} \langle \mathbf{E} \cdot \mathbf{B} \rangle &= - \int \frac{dk}{(2\pi)^2 a^4} J(k) \\ &= - \int \frac{dk k^3}{(2\pi)^2 a^4} \frac{\partial}{\partial \tau} (|A_+|^2 - |A_-|^2). \end{aligned} \quad (18)$$

The integrand in Eq. (18) has a different divergent behavior compared to the energy density, since it has only quadratic and logarithmic divergences,

$$J_{\text{div}}(k) \sim \frac{\xi k}{\tau^2} - \frac{3\xi(1 - 5\xi^2)}{2\tau^4 k} + \mathcal{O}\left(\frac{1}{k}\right)^{5/2}. \quad (19)$$

Also, in this case, the integrand in Eq. (18) does not have any IR pathology. We point out that, even if we called it ‘‘helicity integral,’’ the above integral in Eq. (18) is actually the *derivative* of what is usually called the helicity integral  $\mathcal{H} = \langle \mathbf{A} \cdot \mathbf{B} \rangle$  (which is also gauge invariant for a coupling to a pseudoscalar).

### IV. ANALYTICAL CALCULATION OF DIVERGENT AND FINITE TERMS

In order to find an analytical expression for the finite part, we note that the bare integrals in Eqs. (15) and (18) can be solved by using the expression of the mode functions  $A_\pm$  given in Eq. (9). We identify the divergences by imposing a UV physical cutoff  $\Lambda$  in order to avoid time-dependent coupling constants at low energies: we therefore impose a comoving  $k$ -cutoff  $k_{UV} = \Lambda a$  [37,52] in the integrals in Eqs. (15) and (18). Note that Eq. (7) can be solved analytically by Whittaker functions also in presence of a mass term [22]. We restrict in this section to the results for the massless case, but we also give the generalization to nonzero mass in Appendix B.

### A. Energy density

We first compute the energy density (see Appendix A for details). With the help of the integral representation of the Whittaker functions and carefully choosing the integration contour, we obtain for the energy density stored in the electric field,

$$\begin{aligned} \frac{\langle \mathbf{E}^2 \rangle}{2} = & \frac{\Lambda^4}{16\pi^2} - \frac{\xi^2 H^2}{16\pi^2} \Lambda^2 + \frac{\xi^2 (19 - 5\xi^2) H^4 \log(2\Lambda/H)}{32\pi^2} + \frac{9\gamma H^4 \xi^2 (7 - 5\xi^2)}{32\pi^2} + \frac{H^4 (148\xi^6 - 163\xi^4 - 221\xi^2 + 18)}{96\pi^2 (\xi^2 + 1)} \\ & + \frac{H^4 \xi (30\xi^2 - 11) \sinh(2\pi\xi)}{128\pi^3} - \frac{iH^4 \xi^2 (5\xi^2 - 19) \sinh(2\pi\xi) \psi^{(1)}(1 - i\xi)}{128\pi^3} \\ & + \frac{iH^4 \xi^2 (5\xi^2 - 19) \sinh(2\pi\xi) \psi^{(1)}(i\xi + 1)}{128\pi^3} + \frac{H^4 \xi^2 (5\xi^2 - 19) (H_{-i\xi-2} + H_{i\xi-2})}{64\pi^2}, \end{aligned} \quad (20)$$

while for the magnetic field, we find

$$\begin{aligned} \frac{\langle \mathbf{B}^2 \rangle}{2} = & \frac{\Lambda^4}{16\pi^2} + \frac{3\xi^2 H^2}{16\pi^2} \Lambda^2 + \frac{5\xi^2 (7\xi^2 - 5) H^4 \log(2\Lambda/H)}{16\pi^2} - \frac{H^4 (533\xi^4 - 715\xi^2 + 36)}{384\pi^2} \\ & + \frac{H^4 (210\xi^4 - 185\xi^2 + 18) \sinh(2\pi\xi)}{384\pi^3 \xi} + \frac{5iH^4 \xi^2 (7\xi^2 - 5) \sinh(2\pi\xi) \psi^{(1)}(1 - i\xi)}{128\pi^3} \\ & - \frac{5iH^4 \xi^2 (7\xi^2 - 5) \sinh(2\pi\xi) \psi^{(1)}(i\xi + 1)}{128\pi^3} - \frac{5H^4 \xi^2 (7\xi^2 - 5) [\psi(-i\xi) + \psi(i\xi)]}{64\pi^2}. \end{aligned} \quad (21)$$

Summing the two contributions, the total energy density becomes

$$\begin{aligned} \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} = & \frac{\Lambda^4}{8\pi^2} + \frac{\xi^2 H^2}{8\pi^2} \Lambda^2 + \frac{3\xi^2 (5\xi^2 - 1) H^4 \log(2\Lambda/H)}{16\pi^2} + \frac{\gamma (11 - 10\xi^2) \xi^2 H^4}{8\pi^2} + \frac{\xi (30\xi^2 - 11) \sinh(2\pi\xi) H^4}{64\pi^3} \\ & + \frac{\xi^2 (7\xi^6 - 282\xi^4 + 123\xi^2 + 124) H^4}{256\pi^2 (\xi^2 + 1)} - \frac{3i\xi^2 (5\xi^2 - 1) \sinh(2\pi\xi) \psi^{(1)}(1 - i\xi) H^4}{64\pi^2} \\ & + \frac{3i\xi^2 (5\xi^2 - 1) \sinh(2\pi\xi) \psi^{(1)}(1 + i\xi) H^4}{64\pi^2} + \frac{3\xi^2 (5\xi^2 - 1) [\psi(-i\xi - 1) + \psi(i\xi - 1)] H^4}{32\pi^2}, \end{aligned} \quad (22)$$

where  $\psi$  is the Digamma function,  $\psi^{(1)}(x) \equiv d\psi(x)/dx$ ,  $H_x \equiv \psi(x+1) + \gamma$  is the harmonic number of order  $x$ , and  $\gamma$  is the Euler-Mascheroni constant. The finite terms in Eq. (22) have the corresponding asymptotic behavior,

$$\frac{H^4}{64\pi^2} (21 + 76\gamma) \xi^2 \quad \text{when } |\xi| \ll 1, \quad (23)$$

$$\frac{9H^4 \sinh(2\pi\xi)}{1120\pi^3 \xi^3} \quad \text{when } |\xi| \gg 1. \quad (24)$$

We now compare our results with those used in the literature which are based on the use of UV and IR cutoffs and an approximation of the integrand. More precisely Refs. [1,5] and subsequent works use the following approximation to estimate the integral.

- (i) The integral has a physical UV cutoff at  $-k\tau = 2|\xi|$ .
- (ii) Only the growing mode function  $A_+$  in Eq. (9) is considered for  $\xi > 0$  (the situation is reversed for  $\xi < 0$ ) and it is approximated in this regime to

$$A_+(\tau, k) \simeq \frac{1}{\sqrt{2k}} \left( \frac{-k\tau}{2\xi} \right)^{1/4} e^{\pi\xi - \sqrt{-8\xi k\tau}}. \quad (25)$$

Under this approximation, Ref. [5] obtains when  $\xi \gg 1$ ,

$$\frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\text{AS}}}{2} \simeq 1.4 \times 10^{-4} \frac{H^4}{\xi^3} e^{2\pi\xi}. \quad (26)$$

Our result in Eq. (22) is one of the main original results of this paper and is valid for any  $\xi$ .

In Fig. 1, we plot the terms of Eq. (22), which do not depend on the UV cutoff,

$$\begin{aligned} \mathcal{I}_{\text{fin}}(\xi) \equiv & \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} - \left[ \frac{\Lambda^4}{8\pi^2} + \frac{\xi^2 H^2}{8\pi^2} \Lambda^2 \right. \\ & \left. + \frac{3\xi^2 (5\xi^2 - 1) H^4 \log(2\Lambda/H)}{16\pi^2} \right] \end{aligned} \quad (27)$$

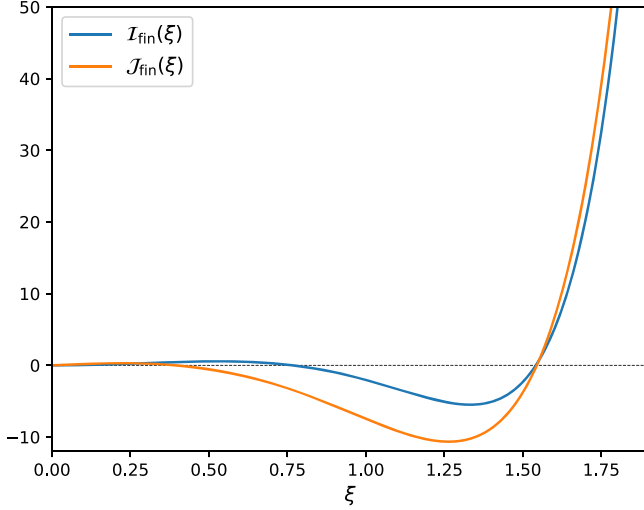


FIG. 1. We plot, respectively, the quantities  $\mathcal{I}_{\text{fin}}(\xi)$  (blue line) and  $\mathcal{J}_{\text{fin}}(\xi)$  (orange line) defined in Eqs. (27) and (33).

in units of  $(2\pi)^2/H^4$  (for  $\xi > 0$  for simplicity). These terms would correspond to the renormalized energy density obtained by a minimal subtraction scheme.

In Fig. 2, we plot the electric and magnetic finite contributions to the energy density (by restricting to  $\xi > 0$ , again). The electric contribution to the energy density is larger than the magnetic one for  $\xi \gtrsim 0.75$ , whereas they are comparable for  $\xi \lesssim 0.75$ .

In Fig. 3, we plot the relative difference with Eq. (26),

$$\Delta\mathcal{I}_{\text{fin}}(\xi) \equiv \frac{2\mathcal{I}_{\text{fin}}(\xi) - \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\text{AS}}}{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\text{AS}}}. \quad (28)$$

It can be seen from Eq. (24) and Fig. 3 that the behavior in the regime  $|\xi| \gg 1$  of our solution is similar

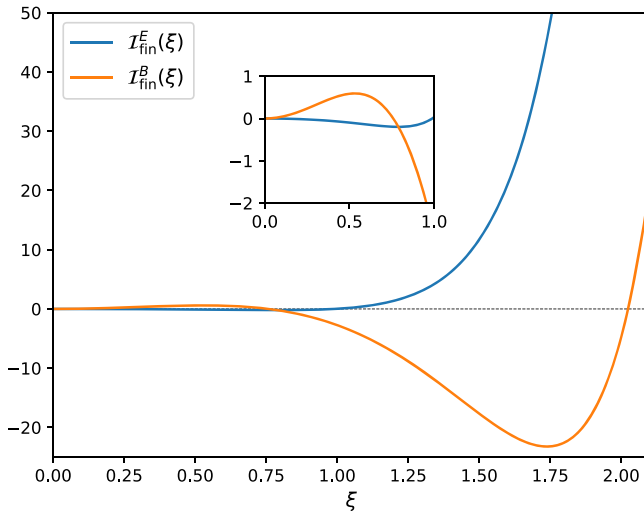


FIG. 2. We plot in blue (orange) the electric (magnetic) contribution to  $\mathcal{I}_{\text{fin}}(\xi)$ .

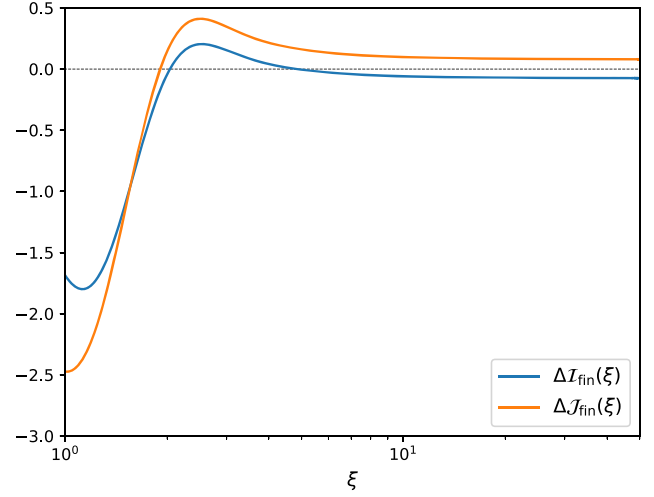


FIG. 3. We plot respectively the quantities  $\Delta\mathcal{I}_{\text{fin}}(\xi)$  (blue line) and  $\Delta\mathcal{J}_{\text{fin}}(\xi)$  (orange line) defined in Eqs. (28) and (34).

to that of Eq. (26), which has thoroughly been studied in the literature. Nevertheless, Fig. 3 shows that there is a relative difference of approximately the order of 10% in the numerical coefficient that multiplies  $\exp(2\pi\xi)/\xi^3$ , which can be ascribed to the assumptions described above.

The main difference of our new result in Eq. (22) with respect to Eq. (26) is in the regime of  $|\xi| \lesssim 10$ , that has been studied for the first time in this paper. Equation (26) cannot be extrapolated to  $|\xi| \ll 1$ , whereas our result shows that the finite part of the energy density is  $\mathcal{O}(\xi^2)$  as shown in Eq. (23). This difference can be understood by noting that the contributions from  $A_+$  and  $A_-$  become comparable in this regime of  $\xi$  and neglecting  $A_-$  is no longer a good approximation.

We end on noting that the finite contribution by a minimal subtraction scheme to the energy density, which is  $\mathcal{O}(\xi^2)$  for  $\xi \ll 1$ , becomes negative for  $0.8 \lesssim \xi \lesssim 1.5$ , although its classical counterpart of Eq. (22) is positive definite. This is not totally surprising since it is known that in QFT in curved space-times the renormalized terms of expectation values of classically defined positive terms can be negative [27].

## B. Helicity integral

The helicity integral in Eq. (18) is only logarithmically and quadratically divergent because of the cancellation of the quartic divergence and does not exhibit any IR divergence. Note that only by considering both  $A_+$  and  $A_-$ , quartic divergent terms in the UV regime cancel.

It is possible to derive an exact solution of Eq. (18) with a UV cutoff and we give the final result in the following, leaving the details for the interested reader in Appendix A. The result for the helicity is

$$\begin{aligned}
-\langle \mathbf{E} \cdot \mathbf{B} \rangle = & \frac{\xi H^2}{8\pi^2} \Lambda^2 + \frac{3\xi(5\xi^2 - 1)H^4 \log(2\Lambda/H)}{8\pi^2} \\
& + \frac{[6\gamma\xi(5\xi^2 - 1) + (22\xi - 47\xi^3)]H^4}{16\pi^2} \\
& + \frac{(30\xi^2 - 11) \sinh(2\pi\xi)H^4}{32\pi^3} \\
& - \frac{3\xi(5\xi^2 - 1)(H_{-i\xi} + H_{i\xi})H^4}{16\pi^2} \\
& + i \frac{3\xi(5\xi^2 - 1) \sinh(2\pi\xi)\psi^{(1)}(1 - i\xi)H^4}{32\pi^3} \\
& - i \frac{3\xi(5\xi^2 - 1) \sinh(2\pi\xi)\psi^{(1)}(i\xi + 1)H^4}{32\pi^3}.
\end{aligned} \tag{29}$$

The finite terms have the corresponding asymptotic values,

$$\frac{H^4}{32\pi^2} (11 - 6\gamma)\xi \quad \text{when } \xi \ll 1, \tag{30}$$

$$\frac{9 \sinh(2\pi\xi)H^4}{560\pi^3 \xi^4} \quad \text{when } \xi \gg 1. \tag{31}$$

The result reported in the literature for the integral in Eq. (18) is derived under the same assumptions discussed in the context of Eq. (26) and is given by [5]

$$-\langle \mathbf{E} \cdot \mathbf{B} \rangle_{\text{AS}} = \frac{H^4}{\xi^4} e^{2\pi\xi}. \tag{32}$$

Again, we define

$$\begin{aligned}
\mathcal{J}_{\text{fin}}(\xi) \equiv & -\langle \mathbf{E} \cdot \mathbf{B} \rangle \\
& - \left[ \frac{\xi H^2}{8\pi^2} \Lambda^2 + \frac{3\xi(5\xi^2 - 1)H^4 \log(2\Lambda/H)}{8\pi^2} \right],
\end{aligned} \tag{33}$$

and the relative differences between our solution and Eq. (32)

$$\Delta \mathcal{J}_{\text{fin}}(\xi) \equiv -\frac{\mathcal{J}_{\text{fin}}(\xi) + \langle \mathbf{E} \cdot \mathbf{B} \rangle_{\text{AS}}}{\langle \mathbf{E} \cdot \mathbf{B} \rangle_{\text{AS}}}, \tag{34}$$

which we plot in Figs. 1 and 3, respectively.

For  $\xi \ll 1$ , the backreaction in Eq. (30) is reminiscent of an extra dissipative term of the type  $\Gamma_{\text{dS}} \dot{\phi}$ . It is interesting to note that the effective  $\Gamma_{\text{dS}}$  in this nearly de Sitter evolution is larger than the perturbative decay rate  $\Gamma = g^2 m_\phi^3 / (64\pi)$  by a factor  $\mathcal{O}(H^3/m_\phi^3)$ .

Analogously to the energy density case, Fig. 3 shows that in the regime of  $\xi \gtrsim 10$  Eqs. (31) and (32) have a similar functional form, but still a 10% difference. Our exact result can be more precisely used for  $\xi \lesssim 10$  and in particular in the  $\xi \lesssim 1$  regime. Note that the difference

between the exact result and the result given in [5] is now larger than in the energy density case and that we have a linear dependence on  $\xi$  for the helicity integral for  $\xi \ll 1$ , in a regime to which the standard result in the literature in Eq. (32) cannot be extrapolated.

## V. ADIABATIC EXPANSION AND REGULARIZATION

The adiabatic regularization method [37,38] relies on the adiabatic, or Wentzel-Kramer-Brillouin (WKB henceforth), expansion of the mode functions  $A_\pm$  solution of Eq. (6). Following the standard adiabatic regularization procedure, we proceed by adding a mass term regulator  $\mu$  to the evolution equations of the two different helicity states obtaining a modified version of Eq. (6),

$$\frac{d^2}{d\tau^2} A_\pm^{\text{WKB}}(k, \tau) + \left( k^2 \mp gk\phi' + \frac{\mu^2}{H^2 \tau^2} \right) A_\pm^{\text{WKB}}(k, \tau) = 0, \tag{35}$$

where the adiabatic mode function solution  $A_\lambda^{\text{WKB}}$  is defined as

$$A_\lambda^{\text{WKB}}(k, \tau) = \frac{1}{\sqrt{2\Omega_\lambda(k, \tau)}} e^{i \int^\tau d\tau' \Omega_\lambda(k, \tau')}, \tag{36}$$

with  $\lambda = \pm$ . The mass term regulator  $\mu$  is inserted to avoid additional IR divergences which are introduced by the adiabatic expansion for massless fields. Inserting the adiabatic solution (36) in Eq. (35), we then obtain the following exact equation for the WKB frequencies  $\Omega_\lambda$ :

$$\Omega_\lambda^2(k, \tau) = \bar{\Omega}_\lambda^2(k, \tau) + \frac{3}{4} \left( \frac{\Omega'_\lambda(k, \tau)}{\Omega_\lambda(k, \tau)} \right)^2 - \frac{\Omega''_\lambda(k, \tau)}{2\Omega_\lambda(k, \tau)}, \tag{37}$$

with

$$\bar{\Omega}_\lambda^2(k, \tau) = \omega^2(k, \tau) + \lambda k g \phi'(\tau) \tag{38}$$

and

$$\omega^2(k, \tau) = k^2 + \mu^2 a^2(\tau). \tag{39}$$

The usual procedure is then to solve Eq. (37) iteratively, introducing an adiabatic parameter  $\epsilon$  assigning a power of  $\epsilon$  to each of the derivative with respect to  $\tau$ . To arrive to order  $2n$ , we have then to do  $n$  iterations. Finally, we have to further Taylor-expand  $\bar{\Omega}_\lambda$  in power of  $\epsilon$  around  $\epsilon = 0$  discarding all the resulting terms of adiabatic order larger than  $2n$  in the final result.

One can then mode expand  $A_\lambda(\mathbf{x}, \tau)$  using the  $n$ th adiabatic order mode functions  $A_\lambda^{(n)}(k, \tau)$  and the associated adiabatic creation and annihilation operators, and then

define the  $n$ th order adiabatic vacuum as  $a_{\lambda,k}^{(n)}|0^{(n)}\rangle = 0$  when  $\tau \rightarrow -\infty$ . In particular, in our case, we have that  $\omega'/\omega \rightarrow 0$  for  $\tau \rightarrow -\infty$ . Thus, in this limit, the adiabatic vacuum defined at any adiabatic order becomes essentially the adiabatic vacuum of infinite order which we call  $|0\rangle_A$ .

The adiabatic regularization is a procedure to remove the UV divergences and consists in subtracting from an expectation value its adiabatic counterpart. In practice, we will proceed by introducing a UV physical cutoff  $\Lambda$  for the mode integral, performing the subtraction, and only after that we will send the cutoff to infinity. Namely, we have

$$\langle \mathcal{O} \rangle_{\text{reg}} = \lim_{\Lambda \rightarrow \infty} [\langle \mathcal{O} \rangle_{\Lambda} - \langle \mathcal{O} \rangle_{A,\Lambda}], \quad (40)$$

where  $\mathcal{O}$  denotes a quadratic operator in the quantum fields, such as the energy density or the helicity. By  $\langle \mathcal{O} \rangle_{\Lambda}$ , we then

mean the bare expectation value of these operators evaluated with a UV cutoff  $\Lambda$ , while by  $\langle \mathcal{O} \rangle_{A,\Lambda}$  we mean the expectation value of their adiabatic counterpart evaluated using the same UV cutoff  $\Lambda$ .

Considering the energy density and the helicity, the bare expectation values are those computed in the previous section for  $\mu = 0$  [see Eqs. (15) and (18)] or their extensions to  $\mu \neq 0$  given in Appendix B. Their adiabatic counterpart is instead given by their corresponding integrals expressed in terms of the WKB mode functions of a given adiabatic order  $n$ . Namely, these are given by Eqs. (15) and (18) where we take the adiabatic solution in Eq. (36) for the mode function using the solution of adiabatic order  $n$  and expanding again up to order  $n$ . In the case under consideration, the fourth order adiabatic expansion is needed in order to remove the UV divergences from the bare integral. We then obtain

$$\begin{aligned} \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{A,\Lambda}}{2} &= \int_c^{\Lambda a} \frac{dk k^2}{(2\pi)^2} \left[ \left( \frac{1}{2\Omega_+} + \frac{1}{2\Omega_-} \right) \omega^2(k, \tau) + \frac{\Omega_-}{2} + \frac{\Omega_+}{2} + \frac{\epsilon^2 \Omega_-''}{8\Omega_-^3} + \frac{\epsilon^2 \Omega_+''}{8\Omega_+^3} \right] \\ &= \int_c^{\Lambda a} \frac{dk}{(2\pi)^2 a^4} \left[ 2\omega + \frac{\epsilon^2 k^4 \xi^2}{\tau^2 \omega^5} + \frac{\epsilon^2 k^2 \mu^2 \xi^2}{H^2 \tau^4 \omega^5} + \frac{\epsilon^4 15k^8 \xi^4}{4\tau^4 \omega^{11}} - \frac{\epsilon^4 3k^8 \xi^2}{4\tau^4 \omega^{11}} \right. \\ &\quad + \frac{\epsilon^4 3\mu^8}{64H^8 \tau^{12} \omega^{11}} + \frac{\epsilon^4 3k^2 \mu^6}{4H^6 \tau^{10} \omega^{11}} - \frac{\epsilon^4 15k^4 \mu^4}{16H^4 \tau^8 \omega^{11}} + \frac{\epsilon^4 19k^2 \mu^6 \xi^2}{8H^6 \tau^{10} \omega^{11}} + \frac{\epsilon^4 \mu^4}{4H^4 \tau^6 \omega^5} \\ &\quad \left. + \frac{\epsilon^4 67k^4 \mu^4 \xi^2}{8H^4 \tau^8 \omega^{11}} + \frac{\epsilon^4 15k^6 \mu^2 \xi^4}{2H^2 \tau^6 \omega^{11}} + \frac{\epsilon^4 21k^6 \mu^2 \xi^2}{4H^2 \tau^6 \omega^{11}} + \frac{\epsilon^4 15k^4 \mu^4 \xi^4}{4H^4 \tau^8 \omega^{11}} \right], \quad (41) \end{aligned}$$

$$\begin{aligned} \langle \mathbf{E} \cdot \mathbf{B} \rangle_{A,\Lambda} &= - \int_c^{\Lambda a} \frac{dk k^2}{(2\pi)^2 a^4} \left[ \frac{\epsilon \Omega_+'}{2\Omega_+^2} - \frac{\epsilon \Omega_-'}{2\Omega_-^2} \right] = \int_c^{\Lambda a} \frac{dk}{(2\pi)^2 a^4} \left[ - \frac{\epsilon^2 k^3 \xi}{\tau^2 \omega^5} + \frac{\epsilon^2 2k\mu^2 \xi}{H^2 \tau^4 \omega^5} + \frac{\epsilon^4 121k^3 \mu^4 \xi}{8H^4 \tau^8 \omega^{11}} \right. \\ &\quad \left. - \frac{\epsilon^4 15k^7 \xi^3}{2\tau^4 \omega^{11}} + \frac{\epsilon^4 3k^7 \xi}{2\tau^4 \omega^{11}} + \frac{\epsilon^4 5k^5 \mu^2 \xi^3}{2H^2 \tau^6 \omega^{11}} - \frac{\epsilon^4 23k^5 \mu^2 \xi}{H^2 \tau^6 \omega^{11}} + \frac{\epsilon^4 k\mu^6 \xi}{4H^6 \tau^{10} \omega^{11}} + \frac{\epsilon^4 10k^3 \mu^4 \xi^3}{H^4 \tau^8 \omega^{11}} \right]. \quad (42) \end{aligned}$$

In the above equations, the IR  $k$ -cutoff  $c$  is also considered as an alternative to the mass term regulator to cure the IR divergences which appear when considering the fourth order adiabatic expansion of a massless field.

As said, the fourth order adiabatic expansion of the mode functions is sufficient to generate the same UV divergences of the bare integrals in Eqs. (16) and (19). However, the fourth order expansion also generates logarithmic IR divergences since gauge fields are massless. One way to avoid the IR divergence is to use the mass term in Eq. (35) as an IR regulator.<sup>2</sup> In this way, we get for the energy density

$$\begin{aligned} \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{A,\Lambda}^\mu}{2} &= \frac{\Lambda^4}{8\pi^2} + \frac{\Lambda^2 \xi^2 H^2}{8\pi^2} + \frac{3H^4 \xi^2 (5\xi^2 - 1) \log(2\Lambda/H)}{16\pi^2} \\ &\quad - \frac{H^4}{480\pi^2} - \frac{H^4 \xi^2 (23\xi^2 - 9)}{16\pi^2} - \frac{3H^4 \xi^2 (5\xi^2 - 1) \log(\frac{\mu}{H})}{16\pi^2}. \quad (43) \end{aligned}$$

Analogously, for the helicity term, we get

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle_{A,\Lambda}^\mu = \frac{\Lambda^2 \xi H^2}{8\pi^2} + \frac{3H^4 \xi (5\xi^2 - 1) \log(\frac{2\Lambda}{H})}{8\pi^2} - \frac{3H^4 \xi (5\xi^2 - 1) \log(\frac{\mu}{H})}{8\pi^2} + \frac{H^4 (19\xi - 56\xi^3)}{16\pi^2}. \quad (44)$$

<sup>2</sup>See Ref. [53] for an interpretation of the mass regulator  $\mu$  in terms of running the coupling constant.



Note that our WKB ansatz correctly reproduces the UV divergences of the energy density and helicity terms. As already known, the fourth order adiabatic expansions leads also to finite terms, including a term with a logarithmic dependence on the effective mass regulator. Let us also comment on the term independent on  $\xi$ , i.e.,  $H^4/(480\pi^2)$  in Eq. (43). This term is generated by the fourth order adiabatic subtraction and is connected to the conformal anomaly. The term we find corresponds to twice the result for a massless conformally coupled scalar field, i.e.,  $H^4/(480\pi^2) = 2 \times H^4/(960\pi^2)$  [27,54], as expected since the two physical states  $A_{\pm}$  behave like two conformally coupled massless scalar fields for  $\xi = 0$ .<sup>3</sup>

An alternative procedure to avoid IR divergences in the adiabatic subtraction is to introduce a time-independent IR cutoff  $k = c$  in the adiabatic integrals. In this way, we obtain for the energy density and helicity term, respectively,

$$\begin{aligned} \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{A,\Lambda}^c}{2} &= \frac{\Lambda^4}{8\pi^2} + \frac{\Lambda^2 \xi^2 H^2}{8\pi^2} \\ &+ \frac{3H^4 \xi^2 (5\xi^2 - 1) \log(2\Lambda/H)}{16\pi^2} \\ &- \frac{3H^4 \xi^2 (5\xi^2 - 1) \log(2c/(aH))}{16aH\pi^2} \\ &- \frac{c^4}{8\pi^2 a^4} - \frac{c^2 \xi^2 H^2}{8a^2 \pi^2}, \end{aligned} \quad (45)$$

$$\begin{aligned} \langle \mathbf{E} \cdot \mathbf{B} \rangle_{A,\Lambda}^c &= \frac{\Lambda^2 \xi H^2}{8\pi^2} + \frac{3H^4 \xi (5\xi^2 - 1) \log(\frac{2\Lambda}{H})}{8\pi^2} \\ &- \frac{c^2 \xi H^2}{8a^2 \pi^2} - \frac{3H^4 \xi (5\xi^2 - 1) \log(\frac{2c}{aH})}{8\pi^2}. \end{aligned} \quad (46)$$

As for the case with the effective mass regulator, the UV divergences of the energy density and helicity terms are also correctly reproduced, although the finite terms are different. Let us note that the term connected to the conformal anomaly is absent, as it comes from the  $k=0$  pole structure of the WKB energy density integrand [27,57]. By comparing Eqs. (45) and (46) with (43) and (44), the terms which do not depend on  $\Lambda$  obtained with the IR cutoff can have the same time dependence of the

<sup>3</sup>It was recently shown in Ref. [55] that in case of photons the standard result  $\langle T \rangle / 4 = -31H^4/(480\pi^2)$  can be obtained by adiabatic regularization only by including Faddeev-Popov fields. On the other hand, it is possible to get the same result without consideration of the Faddeev-Popov ghosts by, first, calculating the photon vacuum polarization in the closed static FLRW (Einstein) universe in which all geometric terms in the trace conformal anomaly become zero and a nonzero average photon energy density arises due to the Casimir effect [56], and then using the known form of the conserved vacuum polarization tensor in a conformally flat space-time that produces the  $R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2$  term in the trace anomaly. Note that the same procedure yields the correct answer for spins  $s = 0, \frac{1}{2}$ , too.

effective mass term by instead considering a comoving IR cutoff  $c = \Lambda_{\text{IR}} a$ . Further quantitative consistency from the two approaches can be obtained by matching  $\mu$  to a physically motivated value for  $\Lambda_{\text{IR}}$  correspondent to the scale at which the WKB approximation breaks down (see [58,59]).

We have seen that the adiabatic regularization applied to the two physical helicity states, because of the massless nature of the gauge field, introduces logarithmic IR divergences. This effect happens in many other contexts; see, e.g., Ref. [60] for a review of IR effects in de Sitter space-time. Logarithmic IR divergences in averaged quantities also appear in the context of the Schwinger effect in de Sitter where they lead to the so-called IR hyperconductivity effect [40]. Furthermore, it has been shown that such logarithmic IR divergences also appear when using other renormalization methods such as point splitting renormalization and thus seem to be generic and not specific of the adiabatic regularization method [61].

### A. Counterterms

We have introduced a fourth order adiabatic expansion which correctly reproduced the UV divergences of the bare quantities. These divergent terms are associated to non-renormalizable derivative interaction counterterms of the pseudoscalar field,

$$\Delta \mathcal{L} = -\frac{\alpha}{4} \nabla^\mu \nabla^\nu \phi \nabla_\mu \nabla_\nu \phi - \frac{\beta}{4} \nabla^\mu \phi \nabla^\nu \phi \nabla_\mu \phi \nabla_\nu \phi, \quad (47)$$

where  $\alpha$  and  $\beta$  are constants of mass dimension  $-2$  and  $-4$ , respectively.

With the new interaction added, the Klein-Gordon equation for the inflaton becomes

$$[\square - \alpha \square^2 + \beta (\nabla \phi)^2 \square] \phi = V_\phi + \frac{g}{4} F^{\mu\nu} \tilde{F}_{\mu\nu}, \quad (48)$$

where  $\square^2 \equiv \nabla^\mu \nabla^\nu \nabla_\mu \nabla_\nu$ . The two additional terms in Eq. (47) lead to the following modification of the energy density:

$$\begin{aligned} T_{00}^{(\phi)} &= \Lambda + \frac{\phi'^2}{2a^2} + V(\phi) + \frac{\alpha}{a^4} (c_1 \phi'^2 + c_2 \phi' \phi^{(3)}) \\ &+ \frac{\beta}{a^4} c_3 \phi^4, \end{aligned} \quad (49)$$

where we have also added a cosmological constant  $\Lambda$ , and the values of the constants  $c_i$ ,  $i = 1, 2, 3$ , are not important for our purposes.

We now isolate the divergences coming from the energy density and the helicity integral using dimensional regularization [62], where, working in a generic  $n$ -dimensional FRLW space-time, the UV divergences show up as poles at  $n = 4$ . This makes clear and explicit the connection between adiabatic expansion and counterterms. We will use in this section results derived in the previous section.

However, we will keep explicit track of derivatives of the pseudoscalar field  $\phi$  here, instead of using the variable  $\xi$ .

The integral measure in  $n$  dimensions is

$$\int \frac{dk k^2}{(2\pi)^3} \rightarrow \int \frac{dk k^{n-2}}{(2\pi)^{n-1}}. \quad (50)$$

### 1. Energy density

As noted in Secs. III and IV, the energy density of the gauge field presents quartic, quadratic, and logarithmic divergences. From the adiabatic expansion of Eq. (15), the term that contributes to the quartic divergence is

$$\int_0^\infty \frac{dk k^{n-2}}{(2\pi)^{n-1} a^4} 2\sqrt{k^2 + \mu^2 a^2} = \frac{\mu^4}{16\pi^2(n-4)} + \dots, \quad (51)$$

where we have retained only the pole at  $n=4$ . Equation (51) shows that the quartic divergence can be absorbed by the cosmological constant counterterm  $\delta\Lambda$ . Similarly, the quadratic divergence comes from

$$\int_0^\infty \frac{dk k^{n-2}}{(2\pi)^{n-1} a^4} \frac{g^2 k^4 \phi'^2}{4(k^2 + \mu^2 a^2)^{5/2}} = \frac{5g^2 \mu^2 \phi'^2}{32\pi^2 a^2(n-4)} + \dots, \quad (52)$$

which shows that we can absorb the quadratic divergence in the field strength counterterm  $\delta Z$ . Finally, the logarithmic divergences come from the terms

$$\begin{aligned} & \int_0^\infty \frac{dk k^{n-2}}{(2\pi)^{n-1} a^4} \left[ \frac{15g^4 k^8 (\phi')^4}{64(k^2 + \mu^2 a^2)^{11/2}} + \frac{g^2 k^8 (\phi'')^2}{16(k^2 + \mu^2 a^2)^{11/2}} \right. \\ & \quad \left. - \frac{g^2 k^8 \phi^{(3)} \phi'}{8(k^2 + \mu^2 a^2)^{11/2}} \right] \\ & = -\frac{15g^4 \phi'^4}{256\pi^2(n-4)a^4} - \frac{g^2 (\phi'')^2}{64\pi^2 a^4(n-4)} + \frac{g^2 \phi^{(3)} \phi'}{32\pi^2 a^4(n-4)}. \end{aligned} \quad (53)$$

The first term can be absorbed in the counterterm  $\delta\beta$ , whereas the second and the third can be absorbed in the  $\delta\alpha$  counterterm.

### 2. Helicity integral

We now consider the divergences in the adiabatic approximation of  $g\langle \mathbf{E} \cdot \mathbf{B} \rangle$ , since this is the term which enters the Klein-Gordon equation (48), to see which are the counterterms needed to absorb them. The helicity integral contains only quadratic and logarithmic divergences.

The quadratic divergence comes from the term

$$\int_0^\infty \frac{dk k^{n-2}}{(2\pi)^{n-1} a^4} \frac{g^2 k^3 \phi''}{2(k^2 + \mu^2 a^2)^{5/2}} = \frac{5g^2 \mu^2 \phi''}{16\pi^2 a^2(n-4)} + \dots, \quad (54)$$

which, again, can be absorbed in the redefinition of the scalar field  $\delta Z$ . Note that the factor of  $a^2$  at the denominator is not a problem since every term with the derivative of the scalar field in the Klein-Gordon equation (48) contains it.

The logarithmic divergence comes instead from the terms

$$\begin{aligned} & \int_0^\infty \frac{dk k^{n-2}}{(2\pi)^{n-1} a^4} \left[ -\frac{15g^4 k^7 \phi'^2 \phi''}{16(k^2 + \mu^2 a^2)^{11/2}} + \frac{g^2 k^7 \phi^{(4)}}{8(k^2 + \mu^2 a^2)^{11/2}} \right] \\ & = \frac{g^4 \phi'^2 \phi''}{4\pi^2 a^4(n-4)} - \frac{g^2 \phi^{(4)}}{32\pi^2 a^4(n-4)} + \dots, \end{aligned} \quad (55)$$

which can be absorbed in the counterterms  $\delta\beta$  and  $\delta\alpha$ , respectively.

## VI. IMPLICATIONS FOR BACKGROUND DYNAMICS

We now consider the implications of our results for the background dynamics. The regularized helicity integral term behaves as an additional effective friction term and slows down the inflaton motion through energy dissipation into gauge fields. The regularized energy-momentum tensor of the gauge field produces an additional contribution to the Friedmann equations.

In order to study backreaction, we introduce the quantity  $\Delta$  which parametrizes the contribution of the gauge fields to the number of  $e$  folds during inflation,

$$\begin{aligned} N & = H \int \frac{d\phi}{\dot{\phi}} \simeq - \int d\phi \frac{3H^2}{V'} \left[ 1 - g \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{3H\dot{\phi}} \right] \\ & \equiv \bar{N}(1 + \Delta), \end{aligned} \quad (56)$$

where in the second equality we have used the Klein-Gordon equation during slow roll and we have defined  $\bar{N}$  as the number of  $e$  folds without taking backreaction into account. For simplicity, we consider the case of a minimal subtraction scheme to avoid the analysis for different values of the IR mass term regulator or cutoff involved in the adiabatic subtraction described in the previous section.

The extreme case with strong dissipation and strong coupling, i.e.,  $3H\dot{\phi} \ll g\langle E \cdot B \rangle$  with  $\xi \gg 1$ , has been the target of the original study in [1]. For  $\xi \gg 1$ , our exact results for the averaged energy-momentum tensor and helicity term differ about 10% from the approximated ones in [1] and therefore we find estimates consistent with [1] at the same level of accuracy.

As we have shown, our results for  $\xi \lesssim 10$  differ from previous ones in the literature. We can give an estimate of the difference in the number of  $e$  folds. As a working example, we use a linear potential  $V(\phi) = \Lambda^4(1 - C|\phi|)$  and we compare our results obtained with those for  $|\xi| \sim 5$  based on the incorrect extrapolation from  $\xi \gg 1$  in Eqs. (26) and (32). Assuming a standard value of  $H \sim 2 \times 10^{-5} M_{\text{pl}}$ , a coupling  $\sim 60$  and  $|\xi| \sim 5$  we obtain

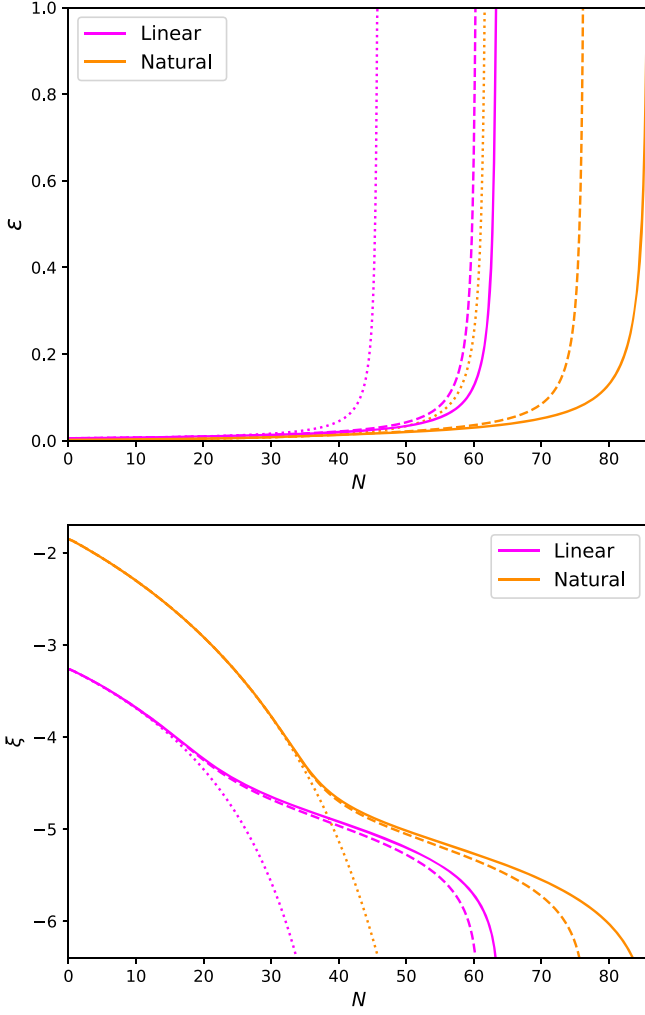


FIG. 4. Numerical evolution of  $-\dot{H}/H^2$  (top) and  $\xi$  (bottom) for the linear model (magenta line) and for natural inflation (orange line) for  $|\xi| \sim 5$  (solid) compared with the case by extrapolating  $\xi \gg 1$  results (dashed) or no coupling (dotted). We have chosen  $|g| = 60/M_{\text{pl}}$  in both cases. For natural inflation, we have used  $f = 5M_{\text{pl}}$ .

$\Delta \simeq 0.32$  and  $0.37$  for the extrapolated and exact result, respectively. Our results thus lead to a 5% longer duration of inflation compared to the extrapolated ones in this case. We note that when backreaction changes the duration of inflation appreciably, it is possible that the gauge field contribution to  $\dot{H}$  is not negligible when observationally relevant scales exit from the Hubble radius, potentially affecting the slopes of the primordial spectra.

To complement and confirm these analytic estimates, we now present numerical results based on the Einstein-Klein-Gordon equations (13), (14), and (17) including the averaged energy-momentum tensor and helicity of gauge fields where we allow  $\xi = g\dot{\phi}/2H$  to vary with time. In the case of the aforementioned linear potential,  $V(\phi) = \Lambda^4(1 - C|\phi|)$ , and of  $V(\phi) = \Lambda^4[1 + \cos(\phi/f)]$  with  $f \sim 2M_{\text{pl}}$  (such value of  $f$  is close to the regime for which

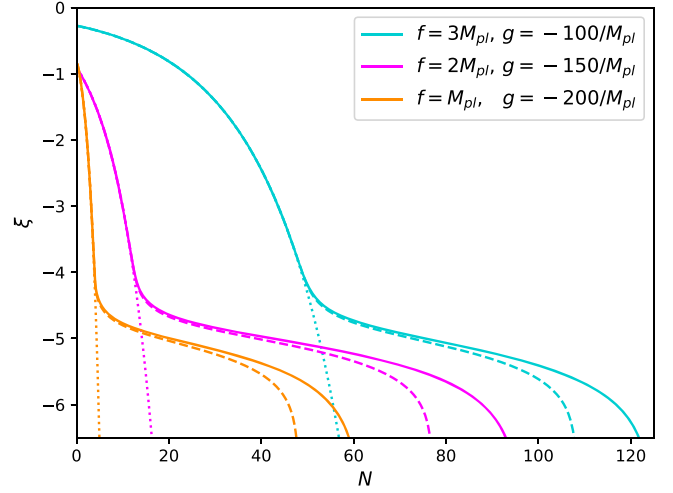


FIG. 5. Numerical evolution of  $\xi$  for natural inflation for three different values of  $f$  and  $g$ . We use dotted lines for no backreaction, dashed lines for extrapolated results for backreaction, and solid lines for our results.

natural inflation is well approximated by a quadratic potential [49]). The results are shown in Fig. 4, comparing our exact results for  $|\xi| \sim 5$  (solid) with those for  $|\xi| \sim 5$  based on the incorrect extrapolation from  $\xi \gg 1$  (dashed) and those in the absence of gauge fields (dotted). As can be seen, the approximation of  $\xi \sim \text{const}$  works very well in both the models.

Figure 5 shows the importance of backreaction in the case of natural inflation for three different values of  $f$ . The inflaton decay into gauge fields allows for a longer period of inflation compared to the case in which coupling to gauge fields is absent. Figures 4 and 5 also show that our correct expressions lead to a longer period of inflation than the incorrect extrapolation from  $\xi \gg 1$ . Furthermore, we show in Fig. 6 how the slow-roll parameter  $\epsilon$  is dominated

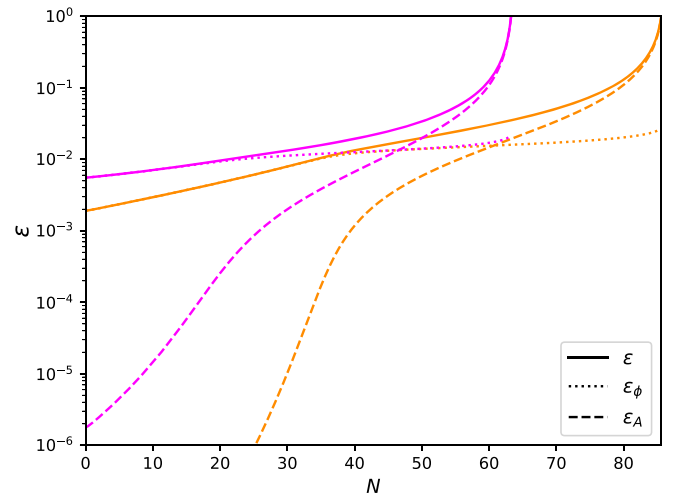


FIG. 6. Numerical evolution of  $\epsilon$ ,  $\epsilon_\phi$ , and  $\epsilon_A$  as defined in the text for the same models shown in Fig. 4.

by  $\epsilon_A \equiv \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle / 3M_{\text{pl}}^2 H^2$  rather than by the usual scalar field contribution  $\epsilon_\phi \equiv \dot{\phi}^2 / 2M_{\text{pl}}^2$  at the end of inflation. Note also that the previously unexplored regime  $\xi \ll 1$  is regular and included in our calculations, whereas the approximation of Eqs. (26) and (32) become singular in this regime.

## VII. CONCLUSIONS

We have studied the backreaction problem for a pseudoscalar field  $\phi$  which drives inflation and is coupled to gauge fields. As in other problems in QFT in curved spacetimes, this backreaction problem is plagued by UV divergences. We have identified the counterterms necessary to heal the UV divergences for this not renormalizable interaction, which are higher order in scalar field derivatives. We have also introduced a suitable adiabatic expansion capable to include the correct divergent terms of the integrated quantities. Under the assumption of a constant time derivative of the inflaton, we have performed analytically the Fourier integrals for the energy density and for the helicity in an exact way with an identification of divergent and finite terms.

Since previous approximate results were available only for  $\xi \gg 1$ , our calculation which is valid for any  $\xi$  has uncovered new aspects of this backreaction problem. We have shown that the regime of validity of previous results is  $\xi \gtrsim 10$  with a 10% level of accuracy. We have then provided results which are more accurate than those present in the literature in the regime  $\xi \lesssim 10$ .

Our results show that the inflaton decay into gauge fields leads to a longer stage of inflation even for  $\xi \lesssim 10$ . This is particularly relevant for natural inflation since  $f \lesssim M_{\text{pl}}$  is a viable regime for a controlled effective field theory [63] and a controlled limit of string theory [64].

The techniques of integration used here in the computation of the bare integrals of  $\langle \mathbf{E}^2 \rangle$ ,  $\langle \mathbf{B}^2 \rangle$ , and  $\langle \mathbf{E} \cdot \mathbf{B} \rangle$  could have a wide range of applications in axion inflation, baryogenesis, and magnetogenesis. Moreover, it would be interesting to compute analytically the energy-momentum tensor and the helicity term for a nonconstant time derivative of the pseudoscalar field which we have adopted in this paper. Other directions would be the calculation of the contribution of gauge fields onto scalar fluctuations leading to non-Gaussian corrections to the primordial power spectra and onto gravitational waves at wavelengths

which range from CMB observations to those relevant for the direct detection from current and future interferometers. We hope to return to these interesting topics in future works.

## ACKNOWLEDGMENTS

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## APPENDIX A: CALCULATION OF THE ENERGY DENSITY AND ITS BACKREACTION

In this appendix, we give the analytical expressions of the bare integrals in Eqs. (15) and (18). We carefully show the calculation of the energy density of Eq. (15); the calculation of Eq. (18) is then straightforward, so we only outline the differences from the one for the energy density. In the following, we will use techniques introduced in Refs. [39,40].

### 1. Energy density

We write Eq. (15) as

$$\frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} \equiv \frac{1}{(2\pi)^2 a^4} \lim_{\Lambda \rightarrow \infty} [\mathcal{I}(\xi, \tau, \Lambda) + \mathcal{I}(-\xi, \tau, \Lambda)], \quad (\text{A1})$$

where

$$\mathcal{I}(\pm\xi, \tau, \Lambda) = \int_0^{\Lambda a} dk k^2 [k^2 (|A_{\pm}|^2) + |A'_{\pm}|^2], \quad (\text{A2})$$

and  $\Lambda$  is a UV physical cutoff (recall that the physical momentum  $k_{\text{phys}}$  is related to the comoving one by  $k_{\text{com}} = k_{\text{phys}} a$ ) used to isolate the UV divergences. Using Eq. (9) and the properties of the Whittaker functions ( $W_{\lambda, \sigma}(x)^* = W_{\lambda^*, \sigma^*}(x^*)$  and  $\frac{d}{dx} W_{\lambda, \sigma}(x) = (\frac{1}{2} - \frac{\lambda}{x}) W_{\lambda, \sigma}(x) - \frac{1}{x} W_{\lambda+1, \sigma}(x)$ ), we obtain

$$\begin{aligned} \mathcal{I}(\xi, \tau, \Lambda) = & \int_0^{\Lambda a} dk k^3 \frac{e^{\pi\xi}}{2} \left\{ \left[ 1 + \left( 1 + \frac{\xi}{k\tau} \right)^2 \right] W_{i\xi, \frac{1}{2}}(-2ik\tau) W_{-i\xi, \frac{1}{2}}(2ik\tau) \right. \\ & + \left( \frac{i}{2k\tau} + \frac{i\xi}{2k^2\tau^2} \right) [W_{i\xi, \frac{1}{2}}(-2ik\tau) W_{-i\xi+1, \frac{1}{2}}(2ik\tau) - W_{-i\xi, \frac{1}{2}}(2ik\tau) W_{i\xi+1, \frac{1}{2}}(-2ik\tau)] \\ & \left. + \frac{1}{2k^2\tau^2} W_{-i\xi+1, \frac{1}{2}}(2ik\tau) W_{i\xi+1, \frac{1}{2}}(-2ik\tau) \right\}. \quad (\text{A3}) \end{aligned}$$

In order to solve this integral, we now make use of the Mellin-Barnes representation of the Whittaker function  $W_{\lambda,\sigma}(x)$ ,

$$W_{\lambda,\sigma}(x) = e^{-\frac{x}{2}} \int_{C_s} \frac{ds}{2\pi i} \frac{\Gamma(-s + \sigma + \frac{1}{2})\Gamma(-s - \sigma + \frac{1}{2})\Gamma(s - \lambda)}{\Gamma(-\lambda - \sigma + \frac{1}{2})\Gamma(-\lambda + \sigma + \frac{1}{2})} x^s, \quad (\text{A4})$$

with  $|\arg x| < \frac{3}{2}\pi$  and the integration contour  $C_s$  runs from  $-i\infty$  to  $+i\infty$  and is chosen to separate the poles of  $\Gamma(-s + \sigma + \frac{1}{2})$  and  $\Gamma(-s - \sigma + \frac{1}{2})$  from those of  $\Gamma(s - \lambda)$ .

Using Eqs. (A3) and (A4), the reflection formula for the gamma functions and integrating the  $k$ -dependent factor up to the cutoff  $\Lambda$ , it is straightforward to find

$$\mathcal{I}(\xi, \tau, \Lambda) + \mathcal{I}(-\xi, \tau, \Lambda) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \quad (\text{A5})$$

where

$$\begin{aligned} \mathcal{I}_1 = & \frac{\sinh^2(\pi\xi)}{2\pi^2} \int_{C_s} \frac{ds}{2\pi i} \int_{C_t} \frac{dt}{2\pi i} (2i\tau)^{s+t} \Gamma(-s)\Gamma(1-s)\Gamma(-t)\Gamma(1-t) \left\{ \left[ (e^{i\pi(s-i\xi)} + e^{i\pi(t+i\xi)}) \frac{(a\Lambda)^{4+s+t}}{4+s+t} \right. \right. \\ & + (e^{i\pi(s-i\xi)} - e^{i\pi(t+i\xi)}) \frac{\xi(a\Lambda)^{3+s+t}}{\tau(3+s+t)} + (e^{i\pi(s-i\xi)} + e^{i\pi(t+i\xi)}) \frac{\xi^2(a\Lambda)^{2+s+t}}{2\tau^2(2+s+t)} \left. \Gamma(s-i\xi)\Gamma(t+i\xi) \right. \\ & + \left. \left[ (e^{i\pi(t-i\xi)} + e^{i\pi(s+i\xi)}) \frac{(a\Lambda)^{4+s+t}}{4+s+t} + (e^{i\pi(t-i\xi)} - e^{i\pi(s+i\xi)}) \frac{\xi(a\Lambda)^{3+s+t}}{\tau(3+s+t)} \right. \right. \\ & \left. \left. + (e^{i\pi(t-i\xi)} + e^{i\pi(s+i\xi)}) \frac{\xi^2(a\Lambda)^{2+s+t}}{2\tau^2(2+s+t)} \right] \Gamma(t-i\xi)\Gamma(s+i\xi) \right\}, \quad (\text{A6}) \end{aligned}$$

$$\begin{aligned} \mathcal{I}_2 = & \frac{\xi \sinh^2(\pi\xi)}{2\pi^2} \int_{C_s} \frac{ds}{2\pi i} \int_{C_t} \frac{dt}{2\pi i} (2i\tau)^{s+t} \Gamma(-s)\Gamma(1-s)\Gamma(-t)\Gamma(1-t) \\ & \times \left\{ (1-i\xi) \left[ (e^{i\pi(t-i\xi)} - e^{i\pi(s+i\xi)}) \frac{1(a\Lambda)^{3+s+t}}{\tau(3+s+t)} + (e^{i\pi(t-i\xi)} + e^{i\pi(s+i\xi)}) \frac{\xi(a\Lambda)^{2+s+t}}{\tau^2(2+s+t)} \right] \Gamma(s+i\xi-1)\Gamma(t-i\xi) \right. \\ & \left. + (1+i\xi) \left[ (e^{i\pi(s-i\xi)} - e^{i\pi(t+i\xi)}) \frac{1(a\Lambda)^{3+s+t}}{\tau(3+s+t)} + (e^{i\pi(s-i\xi)} + e^{i\pi(t+i\xi)}) \frac{\xi(a\Lambda)^{2+s+t}}{\tau^2(2+s+t)} \right] \Gamma(s-i\xi-1)\Gamma(t+i\xi) \right\}, \quad (\text{A7}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}_3 = & \frac{(\xi^2 + \xi^4) \sinh^2(\pi\xi)}{4\pi^2} \int_{C_s} \frac{ds}{2\pi i} \int_{C_t} \frac{dt}{2\pi i} \frac{(a\Lambda)^{2+s+t}}{2+s+t} \frac{1}{\tau^2} (2i\tau)^{s+t} \Gamma(-s)\Gamma(1-s)\Gamma(-t)\Gamma(1-t) \\ & \times [(e^{i\pi(t-i\xi)} + e^{i\pi(s+i\xi)})\Gamma(s+i\xi-1)\Gamma(t-i\xi-1) + (e^{i\pi(s-i\xi)} + e^{i\pi(t+i\xi)})\Gamma(t+i\xi-1)\Gamma(s-i\xi-1)], \quad (\text{A8}) \end{aligned}$$

where we have assumed  $\Re(n+s+t) > 0$  for the terms proportional to  $\Lambda^{n+s+t}$  in order to have convergence.

We now analyze each of these contributions in turn, starting from the first integral in Eq. (A6). We integrate first over the variable  $t$ . Let us further specify the integration contour by requiring  $\Re(t), \Re(s) < 0$ . The integrand can have poles at  $t = \pm i\xi - n$  ( $n = 0, 1, 2, \dots$ ), located on the left of the integration contour of  $t$ , and at  $t = n$  and  $t = -4-s, -3-s, -2-s$ , located on the right of the integration contour of  $t$ . We close the contour counter-clockwise on the left half-plane. The added contours do not contribute to the result since an integral of the integrand over a finite path along the real direction vanishes at

$\Im(t) \rightarrow \pm\infty$  and because any integral in the region  $\Re(t) < -5$  vanishes in the limit  $\Lambda \rightarrow \infty$ . The integral is thus  $2\pi i$  times the sum of the residues of the poles,

$$\begin{aligned} t = & \pm i\xi, \pm i\xi - 1, \pm i\xi - 2, \pm i\xi - 3, \pm i\xi - 4, \pm i\xi - 5, \\ & -s - 4, -s - 3, -s - 2. \quad (\text{A9}) \end{aligned}$$

Note that the poles at  $t = -s-4, -s-3, -s-2$  lie slightly on the right of the axis  $\Re(t) = 0$ , thus we slightly deform the contour to pick it up; the integration contour is illustrated in Fig. 7. The latter poles give a contribution which is independent on the cutoff, whereas the sum of the

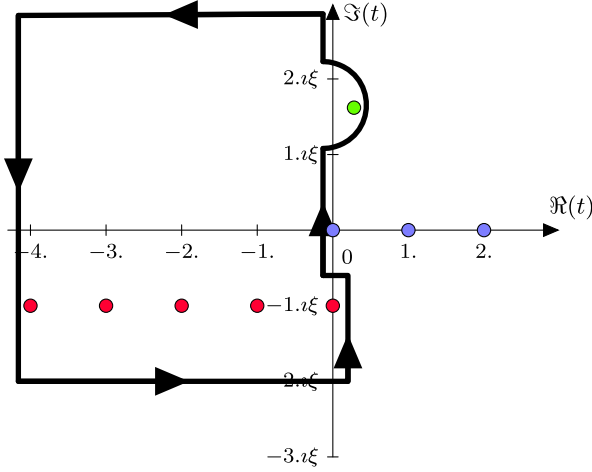


FIG. 7. Integration contour  $C_t$  for the term proportional to  $\Lambda^{4+s+t}\Gamma(t+i\xi)$  of Eq. (A6). Blue points are poles of  $\Gamma(1-t)\Gamma(-t)$  and lie outside the integration contour. Red points are poles of  $\Gamma(t+i\xi)$  and lie inside the contour [in the terms proportional to  $\Gamma(t-i\xi)$  the red points are in  $t=i\xi-n$ ]. The green point is the pole  $t=-s-4$ , and it has been drawn there to emphasize that it is slightly on the right of the imaginary axis (in the terms proportional to  $\Lambda^{3+s+t}$  and  $\Lambda^{2+s+t}$  the green point corresponds to  $t=-s-3, -s-2$ ). The contour does not pass through any of the poles.

former ones gives cutoff-dependent results; we summarize this writing  $\mathcal{I}_1$  as

$$\mathcal{I}_1 = \mathcal{I}_{1,\Lambda} + \mathcal{I}_{1,\text{fin}}. \quad (\text{A10})$$

We first analyze the cutoff-dependent part of the result  $\mathcal{I}_{1,\Lambda}$  that we summarize as follows in order to reduce clutter:

$$\begin{aligned} \mathcal{I}_{1,\Lambda} &= \int_{C_s} \frac{ds}{2\pi i} \Gamma(1-s)\Gamma(-s) \{ \Gamma(s-i\xi) \\ &\quad \times [\mathcal{O}_1(\Lambda^{4+s-i\xi}, \dots, \Lambda^{-1+s-i\xi})] \\ &\quad + \Gamma(s+i\xi) [\mathcal{O}_2(\Lambda^{4+s+i\xi}, \dots, \Lambda^{-1+s+i\xi})] \}. \end{aligned} \quad (\text{A11})$$

The integral over  $s$  can be carried out in the same way as the  $t$  integral and is thus  $2\pi i$  times the sum over the residues of the integrand from the points

$$s = \pm i\xi, \pm i\xi - 1, \pm i\xi - 2, \pm i\xi - 3, \pm i\xi - 4, \quad (\text{A12})$$

the residues from the points  $s = \pm i\xi - n$  with  $n > 4$  vanish as  $\Lambda \rightarrow \infty$ . We schematically write the result of this integral as

$$\begin{aligned} \mathcal{I}_{1,\Lambda} &= f_4(\xi, \tau)\Lambda^4 + f_2(\xi, \tau)\Lambda^2 + f_{\log}(\xi, \tau) \log(2\Lambda/H) \\ &\quad + f_1(\xi, \tau), \end{aligned} \quad (\text{A13})$$

and we will write explicitly only the final result, together with the results from the integral  $\mathcal{I}_2$  and  $\mathcal{I}_3$ .

We now turn to calculate  $\mathcal{I}_{1,\text{fin}}$ , which is the sum of the pole of the integrand in the points  $t=-4-s, -3-s, -2-s$  and can be written as

$$\mathcal{I}_{1,\text{fin}} = \mathcal{I}_{1,t=-4-s} + \mathcal{I}_{1,t=-3-s} + \mathcal{I}_{1,t=-2-s}. \quad (\text{A14})$$

We analyze in detail the integral over the pole  $t=-4-s$ ; the other two are similar. The former is given by

$$\begin{aligned} \mathcal{I}_{1,t=-4-s} &= \int_{C_s} \frac{ds}{2\pi i} \frac{\pi \sinh^2(\pi\xi)}{64\tau^4} \left\{ \frac{(e^{-\pi\xi-i\pi s} + e^{\pi\xi+i\pi s})}{\sin^2(\pi s) \sin(\pi(s-i\xi))} \right. \\ &\quad \left. \times \left[ \frac{a_r}{s-i\xi} + B_r(s) - B_r(s-1) \right] \right\} + \xi \rightarrow -\xi, \end{aligned} \quad (\text{A15})$$

where  $\xi \rightarrow -\xi$  stands for a second integral equal to the first one, but with  $\xi$  replaced by  $-\xi$  and  $B_r(s)$  is a function of the form

$$\begin{aligned} B_r(s) &= \frac{b_{r,1}}{s-i\xi+1} + \frac{b_{r,2}}{s-i\xi+2} + \frac{b_{r,3}}{s-i\xi+3} + \frac{b_{r,4}}{s-i\xi+4} \\ &\quad + b_{r,5} + b_{r,6}s^2 + b_{r,7}s^3 + b_{r,8}s^4, \end{aligned} \quad (\text{A16})$$

and the coefficients  $a_r, b_{r,j}$  for  $j = 1, \dots, 8$  are independent on  $s$ .

We first consider the term with  $a_r$ . We rewrite the integrand as follows:

$$\lim_{p \rightarrow 1} \frac{\pi \sinh^2(\pi\xi)}{64\tau^4} \frac{(e^{-\pi\xi-i\pi s} + e^{\pi\xi+i\pi s})}{\sin^2(\pi s) \sin(\pi(s-i\xi))} \frac{a_r}{(s-i\xi)^p}, \quad (\text{A17})$$

with  $p > 1$ . The integral of this function vanishes on an arc of infinite radius on the left half-plane, so we can close the contour on the left half-plane with a counterclockwise semicircle of infinite radius, as illustrated in Fig. 8. The integral is then  $2\pi i$  times the sum of the residues in the poles  $s = \pm i\xi - n$  and  $s = -n - 1$  with  $n = 0, 1, 2, \dots$ . As for the other integrals, we do not give the result here, but we will just write the final result.

Now we conclude integrating the terms with  $B_r(s) - B_r(s-1)$ . We shift the integration variable in the second term by  $s \rightarrow y = s-1$  so that the integral is given by

$$\begin{aligned} &\int_{C_s} \frac{ds}{2\pi i} (\dots) [B_r(s) - B_r(s-1)] \\ &= \left( \int_{C_s} - \int_{C_{s-1}} \right) \frac{ds}{2\pi i} (\dots) B_r(s), \end{aligned} \quad (\text{A18})$$

as illustrated in Fig. 9. Thus, we can evaluate this integral summing  $2\pi i$  times the residues of the singularities of the integrand which fall in the region sandwiched by the

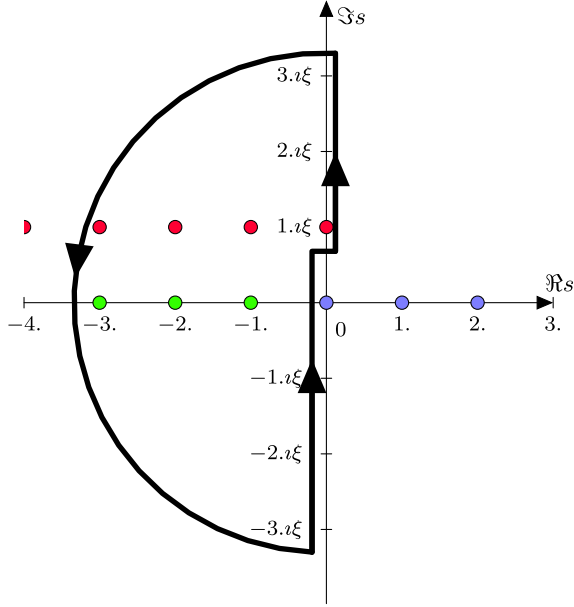


FIG. 8. Integration contour for the term proportional to  $a_r$  in Eq. (A15). The radius of the semicircle is taken to be infinite, and the contour does not pass through any of the poles. Blue points are the poles of  $\Gamma(1-s)\Gamma(-s)$ , whereas red points are the poles of  $\Gamma(s-i\xi)$  and  $\csc(s-i\xi)$ . Green points are the poles of  $\csc(\pi s)$ . For the term with  $\xi \rightarrow -\xi$  in Eq. (A15), the red points move to  $s = -i\xi - n$ .

original integration contour and the shifted one, which are the poles at  $s = -1$  and  $s = \pm i\xi$ . We write the result as  $F_1(\xi, \tau)$ .

The integral over the poles  $t = -s - 3, -s - 2$  can be written in a similar way as

$$\begin{aligned} \mathcal{I}_{1,t=-3-s} &= \int_{C_s} \frac{ds}{2\pi i} \frac{\mathcal{A}_3(\xi, \tau)}{\sin^2(\pi s) \sin(\pi(s-i\xi))} \\ &\times \left[ \frac{c_r}{s-i\xi} + D_r(s) - D_r(s-1) \right] + \xi \rightarrow -\xi \end{aligned} \quad (\text{A19})$$

for the integral over the pole  $t = -3 - s$  and

$$\begin{aligned} \mathcal{I}_{1,t=-2-s} &= \int_{C_s} \frac{ds}{2\pi i} \frac{\mathcal{A}_2(\xi, \tau)}{\sin^2(\pi s) \sin(\pi(s-i\xi))} \\ &\times \left[ \frac{e_r}{s-i\xi} + K_r(s) - K_r(s-1) \right] + \xi \rightarrow -\xi \end{aligned} \quad (\text{A20})$$

for the integral over the pole  $t = -2 - s$ . The functions  $\mathcal{A}_2$  and  $\mathcal{A}_3$  are regular functions of  $\xi$ .  $D_r(s)$  is given by

$$\begin{aligned} D_r(s) &= \frac{d_{r,1}}{s-i\xi+1} + \frac{d_{r,2}}{s-i\xi+2} + \frac{d_{r,3}}{s-i\xi+3} \\ &+ d_{r,4}s + d_{r,5}s^2 + d_{r,6}s^3 \end{aligned} \quad (\text{A21})$$

and  $K_r(s)$  is given by

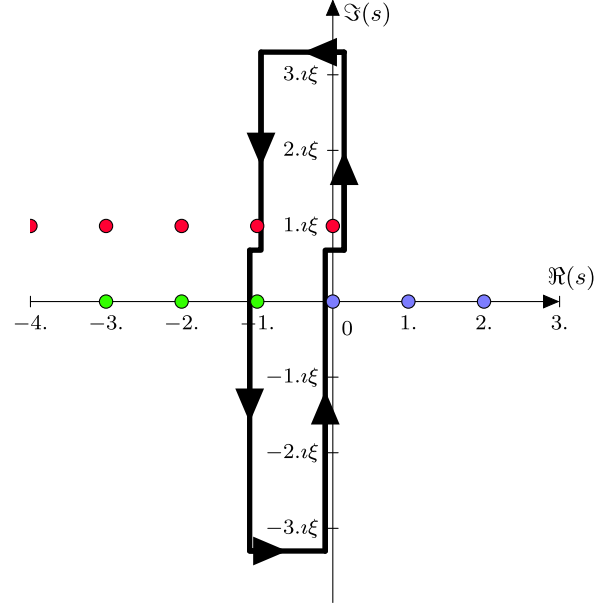


FIG. 9. Integration contour in Eq. (A18). The contour does not pass through any of the poles. Blue points are the poles of  $\Gamma(1-s)\Gamma(-s)$ , whereas red points are the poles of  $\Gamma(s-i\xi)$  and  $\csc(s-i\xi)$ . Green points are the poles of  $\csc(\pi s)$ . For the term with  $\xi \rightarrow -\xi$  in Eq. (A15), the red points move to  $s = -i\xi - n$ .

$$K_r(s) = \frac{k_{r,1}}{s-i\xi+1} + \frac{k_{r,2}}{s-i\xi+2} + k_{r,3}s + k_{r,4}s^2. \quad (\text{A22})$$

The integrals in Eqs. (A19) and (A20) can be made exactly as done in the previous case for the integral over the residue in  $t = -s - 4$  in Eq. (A15). The full solution of the integral in Eq. (A6) is thus

$$\begin{aligned} \mathcal{I}_1 &= f_4(\xi, \tau)\Lambda^4 + f_2(\xi, \tau)\Lambda^2 \\ &+ f_{\log}(\xi, \tau) \log(2\Lambda/H) + f_1(\xi, \tau) + f_{\text{fin}}(\xi, \tau), \end{aligned} \quad (\text{A23})$$

where we have included the contributions of Eqs. (A15), (A19), and (A20) in the term  $f_{\text{fin}}(\xi, \tau)$ .

The integrals  $\mathcal{I}_2$  and  $\mathcal{I}_3$  can be done following the same procedure and we only give the final result in the following.<sup>4</sup>

Defining the contributions to the divergences and finite part of Eq. (A5) as

$$\begin{aligned} \mathcal{I}(\xi, \tau, \Lambda) + \mathcal{I}(-\xi, \tau, \Lambda) &= g_4(\xi, \tau)\Lambda^4 + g_2(\xi, \tau)\Lambda^2 + g_{\log}(\xi, \tau) \log(2\Lambda/H) \\ &+ g_{\text{fin}}(\xi, \tau), \end{aligned} \quad (\text{A24})$$

it can be found that the coefficient of the quartic divergence is

<sup>4</sup>Note that  $\mathcal{I}_2$  is not explicitly invariant under the exchange of  $t$  with  $s$ : we therefore symmetrize it before taking the integral in  $s$  and  $t$ .

$$g_4(\xi, \tau) = \frac{1}{2H^4\tau^4}. \quad (\text{A25}) \quad \text{The coefficient of the logarithmic divergence is}$$

The coefficient of the quadratic divergence is

$$g_{\log}(\xi, \tau) = \frac{3\xi^2(5\xi^2 - 1)}{4\tau^4}. \quad (\text{A27})$$

$$g_2(\xi, \tau) = \frac{\xi^2}{2H^2\tau^4}. \quad (\text{A26})$$

The finite part is

$$g_{\text{fin}}(\xi, \tau) = \frac{3\xi^2(5\xi^2 - 1)(\psi(-i\xi - 1) + \psi(i\xi - 1))}{8\tau^4} + \frac{\gamma(11 - 10\xi^2)\xi^2}{2\tau^4} + \frac{\xi^2(7\xi^6 - 282\xi^4 + 123\xi^2 + 124)}{64(\xi^2 + 1)\tau^4} \\ - \frac{3i\xi^2(5\xi^2 - 1) \sinh(2\pi\xi)(\psi^{(1)}(1 - i\xi) - \psi^{(1)}(i\xi + 1))}{16\pi\tau^4} + \frac{\xi(30\xi^2 - 1) \sinh(2\pi\xi)}{16\pi\tau^4}, \quad (\text{A28})$$

where  $\psi$  is the Digamma function and  $\gamma$  is the Euler-Mascheroni constant.

## 2. BACKREACTION: $\mathbf{E} \cdot \mathbf{B}$

We now calculate the integral in Eq. (18). The quantity we are interested in is

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle = -\frac{1}{(2\pi)^2 a^4} \int dk k^3 \frac{\partial}{\partial \tau} (|A_+|^2 - |A_-|^2) \equiv -\frac{1}{(2\pi)^2 a^4} \lim_{\Lambda \rightarrow \infty} \mathcal{J}(\xi, \tau, \Lambda), \quad (\text{A29})$$

where, obviously,  $\mathcal{J}$  has not to be confused with the one of the previous sections and is given by

$$\mathcal{J}(\xi, \tau, \Lambda) = -\frac{1}{2\tau} \int_0^{\Lambda a} dk k^2 e^{\pi\xi} [W_{i\xi, \frac{1}{2}}(-2ik\tau) W_{-i\xi+1, \frac{1}{2}}(2ik\tau) + W_{-i\xi, \frac{1}{2}}(2ik\tau) W_{i\xi+1, \frac{1}{2}}(-2ik\tau) \\ + W_{-i\xi, \frac{1}{2}}(-2ik\tau) W_{i\xi+1, \frac{1}{2}}(2ik\tau) + W_{i\xi, \frac{1}{2}}(2ik\tau) W_{-i\xi+1, \frac{1}{2}}(-2ik\tau)], \quad (\text{A30})$$

where, again, we put the IR cutoff to 0. As in the previous section, we can use the Mellin-Barnes representation of the Whittaker functions Eq. (A4) to write

$$\mathcal{J}(\xi, \tau, \Lambda) = -\frac{\xi \sinh^2(\pi\xi)}{2\pi^2\tau} \int_{c_s} \frac{ds}{2\pi i} \int_{c_t} \frac{dt}{2\pi i} \frac{\Lambda^{3+s+t}}{3+s+t} (2i\tau)^{s+t} \Gamma(-s)\Gamma(1-s)\Gamma(-t)\Gamma(1-t) \\ \times \{(t+\xi)(e^{i\pi(s+i\xi)} - e^{i\pi(t-i\xi)})\Gamma(s+i\xi-1)\Gamma(t-i\xi) + (-t+\xi)(e^{i\pi(t+i\xi)} - e^{i\pi(s-i\xi)})\Gamma(s-i\xi-1)\Gamma(t+i\xi)\}, \quad (\text{A31})$$

converging for  $\Re(s+t) > -3$ . We only give the final result here since the integral can be carried out as previously explained,

$$\mathcal{J}(\xi, \tau, \Lambda) = \frac{\Lambda^2\xi}{2H^2\tau^4} + \frac{3\xi(5\xi^2 - 1) \log(2\Lambda/H)}{2\tau^4} + \frac{3\gamma\xi(5\xi^2 - 1)}{2\tau^4} + \frac{22\xi - 47\xi^3}{4\tau^4} + \frac{(30\xi^2 - 1) \sinh(2\pi\xi)}{8\pi\tau^4} \\ - \frac{3\xi(5\xi^2 - 1)(H_{-i\xi} + H_{i\xi})}{4\tau^4} + i \frac{3\xi(5\xi^2 - 1) \sinh(2\pi\xi)(\psi^{(1)}(1 - i\xi) - \psi^{(1)}(i\xi + 1))}{8\pi\tau^4}, \quad (\text{A32})$$

where  $H_x = \psi(x+1) + \gamma$  is the Harmonic number of order  $x$  and  $\psi^{(1)}(x) = d\psi(x)/dx$ .

## APPENDIX B: CALCULATION OF THE ENERGY DENSITY AND HELICITY INTEGRAL IN THE PRESENCE OF A MASS TERM

We now generalize the previous calculations to include a mass term. The mode equation for the two helicities, Eq. (7), becomes

$$\left[ \frac{\partial^2}{\partial \tau^2} + k^2 + \frac{\mu^2}{H^2\tau^2} \pm \frac{2k\xi}{\tau} \right] A_{\pm}(k, \tau) = 0, \quad (\text{B1})$$

the solution of which being

$$A_{\pm}(k, \tau) = \frac{1}{\sqrt{2k}} e^{\pm\pi\xi/2} W_{\mp i\xi, \frac{1}{2}} \frac{\mu^2}{H^2} (-2ik\tau). \quad (\text{B2})$$

With this new solution for the mode function, we can compute  $\frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2}$  and  $\langle \mathbf{E} \cdot \mathbf{B} \rangle$  in the same way as done in the previous appendix. There are only two caveats.



- (i) The presence of the mass term modifies the integral in Eq. (18) to

$$\frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} = \int \frac{dk}{(2\pi)^2 a^4} k^2 [k^2 (|A_+|^2 + |A_-|^2) + \mu^2 a^2 (|A_+|^2 + |A_-|^2) + |A'_+|^2 + |A'_-|^2]. \quad (\text{B3})$$

- (ii) In the previous appendix, we expressed the Whittaker functions through their Mellin-Barnes

transform. This led to integrals which we computed closing the contour of integration and using the residue theorem. Among the residues that we have picked up, there were residues from the double poles  $t, s = -n$  where  $n$  was a positive integer number (sometimes  $n = 0$  was included). In the case, where the two helicities acquire a mass, these poles are instead from the single poles  $t, s = -n + \mu^2/H^2$  and  $t, s = -n - \mu^2/H^2$ .

With these caveats in mind, the integrals can be computed leading to

$$\begin{aligned} & \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} \\ &= \frac{\Lambda^4}{8\pi^2} + \frac{H^2 \Lambda^2 \xi^2 (x+1)}{8\pi^2} + \frac{H^4 ((1-x)^2 x^2 + 3\xi^4 (2x+5) + \xi^2 (2(-x-6)(1-x)x - 3)) \log(2\Lambda/H)}{16\pi^2} \\ &+ \frac{H^4 ((x^4 - x^5)(3x-5) - \xi^6 (28x+79) + (x^5(84-x(32x+113)) + 41x^4) + \xi^2 x(x(80-x(4x+45)) + 16) - 8)}{64\pi^2 (\xi^2 + x^2)} \\ &- \frac{H^4 \xi x^2 (x(x(x(2x(x+1)+1) - 684) + 979) + 48) - 192) \sinh(2\pi\xi) \csc(2\pi x)}{1536\pi^2 (\xi^2 + x^2)} \\ &- \frac{H^4 \xi^3 x (x(x(x(8x^2 + 6x - 553) - 1944) + 1286) + 528) \sinh(2\pi\xi) \csc(2\pi x)}{1536\pi^2 (\xi^2 + x^2)} \\ &- \frac{H^8 \xi^2 \sinh(2\pi\xi) \sinh(4\pi\xi) (\xi^2 + x^2)^2 (2\xi^4 + 2x^4 + 2x^3 + \xi^2(4x^2 + 2x + 21) + x^2 - 12x + 19)^2 \cot(2\pi x)}{18874368\pi^4 \sin^2(\pi(x-i\xi)) \sin^2(\pi(x+i\xi)) \sin(2\pi x)} \\ &- \frac{H^4 ((x^2 - x)^2 + 3\xi^4 (2x+5) + \xi^2 ((2x^2 - 2x)(x+6) - 3)) (\psi^{(0)}(-x-i\xi) + \psi^{(0)}(x+i\xi)) (i \sinh(2\pi\xi) \csc(2\pi x) + 1)}{64\pi^2} \\ &+ \frac{H^4 ((x^2 - x)^2 + 3\xi^4 (2x+5) + \xi^2 ((2x^2 - x)(x+6) - 3)) (\psi^{(0)}(x-i\xi) + \psi^{(0)}(-x+i\xi)) (i \sinh(2\pi\xi) \csc(2\pi x) - 1)}{64\pi^2} \end{aligned} \quad (\text{B4})$$

and

$$\begin{aligned} \langle \mathbf{E} \cdot \mathbf{B} \rangle &= \frac{\Lambda^2 \xi H^2}{8\pi^2} + \frac{3H^4 \xi (5\xi^2 - 3(1-x)x - 1) \log(2\Lambda/H)}{8\pi^2} \\ &+ \frac{H^4 (-47\xi^5 - 2\xi^3 (-2(9-17x)x - 11) - \xi x (3 - x(3(10-7x)x + 13)))}{16\pi^4 (\xi^2 + x^2)} \\ &+ \frac{H^4 x \sinh(2\pi\xi) (30\xi^4 + \xi^2 (-2(9-19x)x - 11) + (1-2x)(3-2x)x(2x+1)) \csc(2\pi x)}{16\pi^4 (\xi^2 + x^2)} \\ &- \frac{3iH^4 \xi (5\xi^2 + 3(x-1)x - 1) (\psi^{(0)}(-x-i\xi) + \psi^{(0)}(x+i\xi)) (\sinh(2\pi\xi) \csc(2\pi x) - i)}{32\pi^4} \\ &+ \frac{3iH^4 \xi (5\xi^2 + 3(x-1)x - 1) (\psi^{(0)}(x-i\xi) + \psi^{(0)}(i\xi-x)) (\sinh(2\pi\xi) \csc(2\pi x) + i)}{32\pi^4}, \end{aligned} \quad (\text{B5})$$

where for reasons of convenience we have defined  $x \equiv \mu^2/H^2$ . It can be easily seen that these expressions reduce to those of the previous appendix for  $\mu \rightarrow 0$ .

We note that the presence of a mass term can come for an interaction of the form

$$\mathcal{L} \supset -B(\phi) \frac{F_{\mu\nu} F^{\mu\nu}}{4}, \quad (\text{B6})$$

where the function  $B(\phi)$  is nearly constant in order to break only slightly the shift symmetry breaking of the pseudoscalar

field [22]. This interaction term gives the two helicities an effective mass of the form  $\mu^2 = H^2 \sqrt{\epsilon_\phi \epsilon_B} \text{sign}(B' \dot{\phi})$ , where  $\epsilon_B \equiv \frac{M_{\text{pl}}^2}{2} \left(\frac{B'}{B}\right)^2 \ll 1$  and  $B' = dB/d\phi$ . In this case, the mode function satisfying Eq. (B1) is  $\tilde{A}_\pm(k, \tau) \equiv \sqrt{B(\phi(\tau))} A_\pm(k, \tau)$  and the solution for  $A_\pm(k, \tau)$  is [22]

$$A_\pm(k, \tau) = \frac{1}{\sqrt{2kB_*}} e^{\pm\pi\xi/2} W_{\mp i\xi, \frac{1}{2} + \sqrt{\epsilon_\phi \epsilon_B}}(-2ik\tau), \quad (\text{B7})$$

where  $B_*$  is the value of  $B$  evaluated at horizon crossing for cosmologically interesting modes.

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