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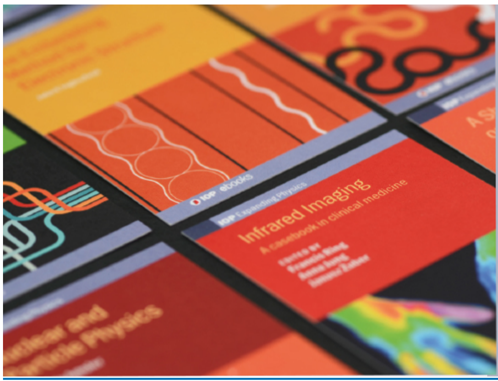

The poor man's magnetohydrodynamic (PMMHD) equations

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Abstract. We present a mathematical derivation of a discrete dynamical system by following a Fourier-Galerkin approximation of the 3-D incompressible magnetohydrodynamic (MHD) equations. In this way, a 6-D map, depending on 12 bifurcation parameters, is derived as a truncated set of nonlinear ordinary differential equations (ODEs) to characterize incompressible plasma dynamical behaviors, also conserving total energy and cross-helicity in the ideal MHD approximation. Moreover, three different subspaces, associated with long-living non-trivial solutions (e.g., fixed point solutions), have been found like the fluid, magnetic, and the Alfvénic fixed points. Our set can be seen as a Lorenz-like model to investigate MHD phenomena.

1. Introduction

Magnetohydrodynamic (MHD) equations allow to describe the macroscopic properties of plasma systems like large-scale structures, waves, as well as, turbulent and intermittent features [e.g., 1–3]. A fascinating question in MHD is concerned with the possibility in "easily" gaining information about plasma dynamics for any given initial condition, i.e., in predicting the final state by looking at the phase-space evolution. This is the main focus of the dynamical systems theory, describing the long-term behavior of complex systems [4]. Dealing with chaotic systems, the focus is not in finding precise solutions, e.g., deterministic solutions, but rather to investigate if there are steady states, if the system will move towards these steady states, and how its long-term behavior depends on initial conditions, e.g., finding a qualitative long-term behavior [5].

With the term *dynamical system* we mean a deterministic mathematical *rule* describing the time-evolution of system state variables $\mathbf{x}(t)$ [4] as

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}, \{\mu\}), \qquad \text{for a } \textit{continuous} \text{ system}, \tag{1}$$

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \{\mu\}), \qquad \text{for a } \textit{discrete} \text{ system or } \textit{map}, \tag{2}$$

being $\mathbf{x}(t)$ (or $\mathbf{x}_n$) an N-dimensional vector whose evolution is only depending on the function $\mathbf{F}$, on the choice of the initial condition $\mathbf{x}(t_0)$ (or $\mathbf{x}_0$), and on the set of bifurcation parameters $\{\mu\}$, such that given $\mathbf{x}(t_0)$ (or $\mathbf{x}_0$) by means of $\mathbf{F}$ we can determine all its forward positions, i.e., a collection of dynamical states known as *trajectory* or *orbit*. One of the most important concepts in dynamical systems theory is that of *fixed point* or steady state which can be identified as the stationary solution $\mathbf{x}^*$ (or $\mathbf{x}_n^*$) of Eqs. (1)-(2)

$$\mathbf{F}(\mathbf{x}^*, \{\mu\}) = 0, \quad (3)$$
$$\mathbf{F}(\mathbf{x}_n^*, \{\mu\}) = 0, \quad (4)$$

whose nature can be characterized by looking at the eigenvalues λ_k of the Jacobian matrix $\mathbb{J}$ associated with the dynamical system, e.g., $\mathbb{J}_{lm} = \frac{\partial \dot{x}_l}{\partial x_m}$, and more specifically by looking at the sign of three different quantities characterizing $\mathbb{J}$ as its determinant $\mathrm{Det}(\mathbb{J}) = \Pi_k \lambda_k$, its trace $\mathrm{Tr}(\mathbb{J}) = \sum_k \lambda_k$, and its discriminant $\Delta(\mathbb{J}) = \mathrm{Tr}(\mathbb{J})^2 - 4\,\mathrm{Det}(\mathbb{J})$.

Different physical systems belong to the class of chaotic systems like the well-known Lorenz system [6], a low-order model for atmospheric convection, the Hénon map [7], a simplified model of the Poincaré section of the Lorenz system, the energy-balance climate models [e.g., 8–10], simplified numerical models for investigating climate transitions among different steady states, as well as, turbulence in fluid flows and MHD systems [11; 12]. In the latter context, Frisch [13] firstly introduced the concept of "poor man's Navier-Stokes (PMNS) equation" to identify a toy model for Navier-Stokes equation based on a logistic map as $u_{n+1} = 1 - 2u_n^2$, being u the flow speed. Then, both 2-D and 3-D extensions of the PMNS equation have been proposed to investigate incompressible fluids properties [14], the emergence of different dynamical behaviors [15], and the transition to chaos [16] when bifurcation parameters are changed.

Here, we derive a 6-D discrete map directly from MHD equations which depends on 12 bifurcation parameters of physical meaning and characterizing the dynamical behaviors of plasma systems by following a Lorenz-like approach [6]. We also identify three different subspaces, corresponding to three different steady-state solutions of MHD equations and associated with non-trivial solutions, namely the fluid, the magnetic, and the Alfvénic solutions, often encountered in natural and laboratory plasmas.

2. The poor man's magnetohydrodynamic (PMMHD) equations

Magnetohydrodynamics is particularly suitable for describing plasma in a quasi-neutral single-fluid approximation, implying that characteristic scales are much larger than the collision ones [17]. In this way, moving from the exact microscopic description (e.g., Klimontovich equation), passing through kinetic (e.g., Vlasov, Landau, Boltzmann, Lenard-Balescu approximations) and multi-fluid descriptions, we can derive a single-fluid description, i.e., the well-known MHD approximation [17]. In the incompressible limit, MHD equations can be written as

$$\partial_t \mathbf{Z}^\pm + \left(\mathbf{Z}^\mp \cdot \nabla\right) \mathbf{Z}^\pm = -\nabla P_{\mathrm{tot}} + \nu_+ \nabla^2 \mathbf{Z}^+ + \nu_- \nabla^2 \mathbf{Z}^- \quad (5)$$
$$\nabla \cdot \mathbf{Z}^\pm = 0 \quad (6)$$

where

- $\mathbf{Z}^\pm \doteq \mathbf{U} \pm \frac{\mathbf{B}}{\sqrt{4\pi\rho_0}}$ are the Elsässer variables [18], being $\mathbf{U}$ and $\mathbf{B}$ the velocity and magnetic fields, respectively, and ρ_0 the plasma mass density;
- $P_{\mathrm{tot}} = \frac{P}{\rho_0} + \frac{B^2}{8\pi\rho_0}$ is the total plasma pressure, i.e., the sum between the kinetic and the magnetic pressure (in terms of the plasma mass density ρ_0);
- $\nu_\pm = \frac{\nu \pm \eta}{2}$ are the pseudo-viscosities, being ν and η the kinematic viscosity and magnetic diffusivity.

Eqs. (5) can be solved by coupling them with a thermodynamic equation or by eliminating the pressure term through a linear operator $\mathbb{L}(\mathbf{V})$ (being $\mathbf{V}$ a vector field) known as Leray operator

$$\mathbb{L}(\mathbf{V}) \doteq \mathbf{V} - \nabla\left[\nabla^{-2}\left(\nabla \cdot \mathbf{V}\right)\right], \quad (7)$$

having two interesting properties:

(i) $\mathbb{L}(\mathbf{V}) = \mathbf{V}$ if $\nabla \cdot \mathbf{V} = 0$,

(ii) $\mathbb{L}(\nabla f) = 0$, being f a scalar function.

Due to property (i) all terms containing $\mathbf{Z}^{\pm}$ in Eqs. (5) remain unchanged, since $\nabla \cdot \mathbf{Z}^{\pm} = 0$, while thanks to property (ii) the pressure gradient vanishes ($\mathbb{L}(\nabla P_{tot}) = 0$, since P_{tot} is a scalar function). Thus, by introducing a dimensionless form we can write Eqs. (5) as

$$\partial_t \mathbf{z}^{\pm} + \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} = \frac{1}{R_{e_+}} \nabla^2 \mathbf{z}^{+} + \frac{1}{R_{e_-}} \nabla^2 \mathbf{z}^{-} \tag{8}$$

where Elsässer variables have been dimensioned to $U_0 + C_A$, being U_0 and C_A the typical flow and Alfvén speed (e.g., $C_A = \frac{B_0}{\sqrt{4\pi\rho_0}}$), respectively, while time and lengths to a typical time $\tau_{\pm}$ and length $L_{\pm}$, respectively. In this way, we can introduce the Reynolds numbers $R_{e_{\pm}} \doteq \frac{(U_0+C_A)L_{\pm}}{\nu_{\pm}}$ which, as expected from the first experiment by Reynolds [19], are the only control parameters of our equations.

Following the seminal work by Lorenz [6], we can derive a set of ordinary differential equations (ODEs) from the set of Eqs. (8), i.e., we can obtain a low-order dynamical system, by considering a truncated Galerkin expansion [6]. Given a vector field $\mathbf{V}(\mathbf{x}, t) \in L^2$, being L^2 a Hilbert space, we can write

$$\mathbf{V}(\mathbf{x}, t) = \sum_{k=-\infty}^{\infty} \mathbf{V}_k(t) \Phi_k(\mathbf{x}), \tag{9}$$

where $\mathbf{V}_k(t)$ is the $k-$th Galerkin coefficient, while $\{\Phi_k(\mathbf{x})\}$ is a complete orthonormal basis through which the Galerkin triple product can be defined as $G_{klm} \doteq (l+m) \langle \Phi_k(\mathbf{x}) \Phi_l(\mathbf{x}) \Phi_m(\mathbf{x}) \rangle$, being $\langle \dots \rangle$ the inner product defined onto the L^2 Hilbert space. Unless considering a Galerkin truncation by selecting a subset of the wavenumbers k such that $k < k^*$ (i.e., by deriving a shell model [11; 20]) or considering a fixed triad-interaction model [e.g., 21], we move towards a dynamical system approach as in Lorenz [6] by retaining only a fixed wavevector [see also 14; 16].

Thus, after some algebra we derive the poor man's MHD (PMMHD) equations

$$z_i^{\pm\prime} = \frac{\beta_i \pm \alpha_i}{2} z_i^{+} + \frac{\beta_i \mp \alpha_i}{2} z_i^{-} + \Gamma_i^j z_i^{\pm} z^{\mp j}, \tag{10}$$

a discrete dynamical system (i.e., a map) as in Frisch [13] in which for the sake of notation simplicity $z_i^{\pm\prime} = z_i^{\pm}(t_{n+1})$ and $z_i^{\pm} = z_i^{\pm}(t_n)$, being $i = 1, 2, 3$. We introduced the bifurcation parameters β_i and α_i defined as

$$\beta_i = 1 - \frac{k^2}{R_e} \frac{L}{\ell_i}, \quad \text{being} \quad R_e = \frac{U_0 L}{\nu}, \tag{11}$$

$$\alpha_i = 1 - \frac{k^2}{R_m} \frac{L}{\ell_i}, \quad \text{being} \quad R_m = \frac{C_A L}{\eta}, \tag{12}$$

depending on the so-called Taylor microscale ℓ_i falling in between the large scales L (i.e., the integral scale) and the small scales η_K (i.e., the Kolmogorov length or microscale) [22]. Moreover, we also introduced a negative third-order coupling tensor Γ_i^j, a symmetric matrix parametrizing the space derivatives of Elsässer variables (e.g., $\frac{\partial z_i^{\pm}}{\partial x_j}$).

3. Rugged invariants

The dynamical behavior of the system can be investigated by looking at the existence of rugged invariants as usual for MHD equations [17; 21]. This can be done by introducing a more compact form of Eqs. (10) as

$$\Psi_i' = M\Psi_i + \mathbf{NL}[\Psi_i, \Psi_j] \tag{13}$$

being $\{\Psi_i\} \doteq \{z_1^+, z_2^+, z_3^+, z_1^-, z_2^-, z_3^-\}$, M a matrix of the dissipative term depending on the bifurcation parameters β_i, α_i, and $\mathbf{NL}[\Psi_i, \Psi_j]$ a short-hand notation for the nonlinear term depending on the bifurcation parameters Γ_i^j.

It is very simple to prove that, in the inviscid form ($\nu = \eta = \nu_\pm = 0 \Rightarrow R_e, R_m, R_{e_\pm} \to \infty$), i.e., when $\beta_i = \alpha_i = 1$, the map admits all rugged invariants of the MHD equations, i.e.,

$$E = \frac{1}{2}\sum_{i=1}^{3} u_i^2 + b_i^2, \quad H_C = \frac{1}{2}\sum_{i=1}^{3} u_i b_i, \tag{14}$$

which, in terms of Elsässer variables correspond to the conservation of both of the pseudo-energies

$$E^+ = \frac{1}{2}\sum_{i=1}^{3} \Psi_i^2, \quad E^- = \frac{1}{2}\sum_{i=4}^{6} \Psi_i^2. \tag{15}$$

Indeed, in the inviscid limit we can write Eqs. (13) as

$$\Psi_i' = \Psi_i + \mathbf{NL}[\Psi_i, \Psi_j] \tag{16}$$

that is a discrete Galerkin approximation of $\frac{D\Psi_i}{Dt} = 0$, being D the Lagrangian derivative, implying the conservation of the pseudo-energies $E^\pm$ [e.g., 17; 21].

4. Fixed points and steady states

As usual in dynamical systems theory, we investigate the existence of steady states or fixed point solutions as described in Sect. 1 for which $\mathbb{T}[\Psi^*] = \Psi^*$, being $\mathbb{T} : \Psi \to \Psi'$. In the following we are looking for fixed points of the usual MHD equations [e.g., 21].

4.1. The 1-D fluid fixed point

We start our investigation by looking at the 1-D case in order to find an analogy with the result obtained by Frisch [13]. The 1-D case can be simply obtained by assuming $z_i^\pm = z^\pm$, implying that $\mathbf{NL}[\Psi_i, \Psi_j] = \mathbf{NL}[z_i^\pm, z_j^\mp] = \mathbf{NL}[z_i^\pm, z_i^\mp]$, providing

$$z^{\pm\prime} = \frac{\beta+\alpha}{2} z^\pm + \frac{\beta-\alpha}{2} z^\mp - \Gamma z^{\pm 2}. \tag{17}$$

By looking for fixed points, i.e., $z^{\pm\prime} = z^\pm = z^{\pm *}$ we reduce to

$$\Gamma z^{\pm * 2} + \left(1 - \frac{\beta+\alpha}{2}\right) z^{\pm *} - \frac{\beta-\alpha}{2} z^{\mp *} = 0, \tag{18}$$

which admits the trivial solution $z^{\pm *} = z^{\mp *}$ reducing to $u^* \pm b^* = u^* \mp b^*$ with the consequence of $b^* = 0$. This means that $z^{\pm *} = u^*$ and we obtain

$$u'^* = \beta u^* - \Gamma u^{*2}, \tag{19}$$

i.e., a logistic evolution for the velocity field which represents a generalized version of the usual 1-D PMNS fluid case [13]. This map admits two fixed points: $u^{*(1)} = 0$ and $u^{*(2)} = \frac{\beta - 1}{\Gamma}$, whose corresponding eigenvalues are $\lambda^{(1)} = \beta$ and $\lambda^{(2)} = 2 - \beta$. Since by definition $\beta \le 1$, we obtain that $u^{*(1)}$ is a stable fixed point ($|\lambda^{(1)}| \le 1$), while $u^{*(2)}$ is an unstable one ($|\lambda^{(2)}| \ge 1$).

4.2. The 3-D fluid fixed point

MHD equations admit a fixed point which is a "fluid" fixed point, corresponding to $z_i^{\pm} = u_i$ for which we trivially obtain the fluid map

$$u_i' = \beta_i u_i + \Gamma_i^j u_i u^j, \tag{20}$$

that is, we reduce to the fluid map obtained by McDonough [16]. By searching for a fixed point $u_i' = u_i = u_i^*$ we note that there exist two fixed points: one is $u_i^{*(1)} = 0$, the other is obtained by solving

$$\Gamma_i u_i^* - \gamma_i^j u_j^* = \beta_i - 1, \tag{21}$$

where we assumed that $\Gamma_i^j = -\Gamma_i \delta_i^j + \gamma_i^j$, being δ_i^j the third-order identity matrix. The first one is a stable fixed point as for the 1-D case since $\lambda_i^{(1)} = \beta_i$ with $|\lambda_i^{(1)}| < 1$, while the stability of the other fixed point depends on the particular choice of bifurcation parameters, moving from a node to a saddle point [16].

4.3. The 3-D magnetic fixed point

The magnetic fixed point of MHD equations can be recovered when $u_i = 0$ which obviously implies that $z_i^{\pm} = \pm b_i$ such that we obtain

$$b_i' = \alpha_i b_i \mp \Gamma_i^j b_i b^j \tag{22}$$

which assumes a similar form of Eq. (20). This obviously leads to similar investigations like the 3-D fluid fixed point according to which there exist two fixed points: one is $b_i^{*(1)} = 0$, which is a stable fixed point, while the other one is obtained by solving

$$\Gamma_i b_i^* \pm \gamma_i^j b_j^* = \alpha_i - 1 \tag{23}$$

whose stability depends on bifurcation parameters $\alpha_i, \Gamma_i, \gamma_i^j$.

4.4. The Alfvénic fixed points

The so-called "Alfvénic" fixed points [21] are defined as $u_i = \pm b_i$ or in terms of Elsässer variables

$$\text{(i) } A^+ : z_i^+ \neq 0, z_i^- = 0, \quad \text{(ii) } A^- : z_i^+ = 0, z_i^- \neq 0. \tag{24}$$

This means Eqs. (10) for both cases reduce to

$$z_i^{\pm\prime} = \frac{\beta_i + \alpha_i}{2} z_i^{\pm} \tag{25}$$

whose fixed points are stable only when $\frac{\beta_i + \alpha_i}{2} = 1$. In the inviscid limit, i.e., when $\beta_i, \alpha_i \to 1$, this fixed point is stable and it exactly corresponds to the MHD case of an Alfvénic perturbation [21].

5. Discussion and conclusions

In this work, we have derived and investigated a discrete dynamical system from three-dimensional incompressible plasma equations. The model can be viewed as the simplest way to investigate complex time behaviors of velocity and magnetic fields in the fluid-like (MHD) approximation of a plasma system. Three different fixed points have been obtained, corresponding to different dynamical situations, i.e., the fluid case, the magnetic one, and the "Alfvénic" points, when bifurcation parameters are changed. The first can be seen as a 3D extension of the well-known fixed point obtained by Frisch [13] and corresponding to the hydrodynamic case; the second relates to magnetic equilibria solutions; finally, the Alfvénic points describe the occurrence of an Alfvénic perturbation coupling z^+ and z^- via the dissipative coefficients ν and η. The PMMHD equation can be seen as a Lorenz-like model which could help in investigating different features and regimes as bifurcation parameters $\beta_i, \alpha_i, \Gamma_i^j$ are changed. Clearly, this model, due to its truncated nature, cannot be used to investigate plasma dynamics inside the MHD domain at all, as the multi-scale behavior and mode-coupling phenomena. Notwithstanding, we think that our model is surely useful to understand the main features and evolution of a MHD plasma system since it shares many properties of the full set of MHD equations. Particularly, with a suitable choice of model parameters we are able in investigating different dynamical behaviors like the Alfvénic solution which can be used to simply gain new insights on alfvénic structures propagation. This means to deal with the problem of the dissipative relaxation phenomena in MHD plasmas, being characterized by a non-trivial temporal evolution of ideal invariants towards a minimum energy state known as Taylor's vortex [21]. Further investigations will be devoted to the characterization of the above behaviors.

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