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# Looking below the floor: constraints on the AGN radio luminosity functions at low power

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## ABSTRACT

We constrain the behaviour of the radio luminosity function (RLF) of two classes of active galactic nuclei (AGN) namely AGN of low radio power (LRP) and BL Lac objects. The extrapolation of the observed steep RLFs to low power predicts a space density of such objects that exceeds that of the sources that can harbour them and this requires a break to a shallower slope. For LRP AGN, we obtain  $P_{\text{br,LRP}} \gtrsim 10^{20.5} \text{ W Hz}^{-1}$  at 1.4 GHz to limit their density to be smaller than that of elliptical galaxies with black hole masses  $M_{\text{BH}} > 10^{7.5} M_{\odot}$ . By combining this value with the limit derived by the observations the break must occur at  $P_{\text{br,LRP}} \sim 10^{20.5} - 10^{21.5} \text{ W Hz}^{-1}$ . For BL Lacs, we find  $P_{\text{br,BLLAC}} \gtrsim 10^{23.3} \text{ W Hz}^{-1}$  otherwise they would outnumber the density of weak-lined and compact radio sources, while the observations indicate  $P_{\text{br,BLLAC}} \lesssim 10^{24.5} \text{ W Hz}^{-1}$ . In the framework of the AGN unified model, a low luminosity break in the RLF of LRP AGN must correspond to a break in the RLF of BL Lacs. The ratio between  $P_{\text{br,LRP}}$  and  $P_{\text{br,BLLAC}}$  is  $\sim 10^3$ , as expected for a jet Doppler factor of  $\sim 10$ .

**Key words:** galaxies: active – BL Lacertae objects: general – galaxies: jets.

## 1 INTRODUCTION

Many astrophysical quantities obey a distribution well described by power laws. Important insights into the physical processes producing such distributions can be obtained from their power-law index. In addition, the location at which the distribution departs from a power law is also of great importance. In most cases, a change in the distribution laws is observed at their high end, in the form of either a steepening of the distribution, or an exponential cutoff. For example, the peak frequency in the synchrotron emission of active galaxies is associated with the largest energy attainable by relativistic electrons and it carries essential constraints on the acceleration and cooling mechanisms. Equally important is the determination of the behaviour of the distributions at the low end as it might reveal the presence of, e.g. a minimum mass for the formation of a star or a galaxy. Unfortunately, the low-end behaviour is often inaccessible to observations or, in other cases, it is blurred by the emergence of strong selection biases.

In this Letter, we focus on the radio luminosity function (RLF) of two classes of active galaxies at low redshift,  $z < 0.1$ . The first is formed by low radio power (LRP) active galactic nuclei (AGN): within the considered redshift limit, most of the radio emitting AGN are indeed objects of low power,  $\log(P_r/\text{W Hz}^{-1}) \lesssim 24$ . The second consists of BL Lac objects. According to the unified model for AGN,

these two RLFs are expected to be connected with each other, as LRP AGN should represent the parent population of BL Lac objects. In both cases, the RLFs are observationally well defined over a very broad range of radio power. The RLF of LRP AGN also presents a break at high luminosity. Conversely, at their low-luminosity end, both RLFs are well described by a pure power law by current observations, without clear signs of a change in their slopes. In this Letter, we show that it is none the less possible to constrain the power at which a low-luminosity break in the RLF must occur.

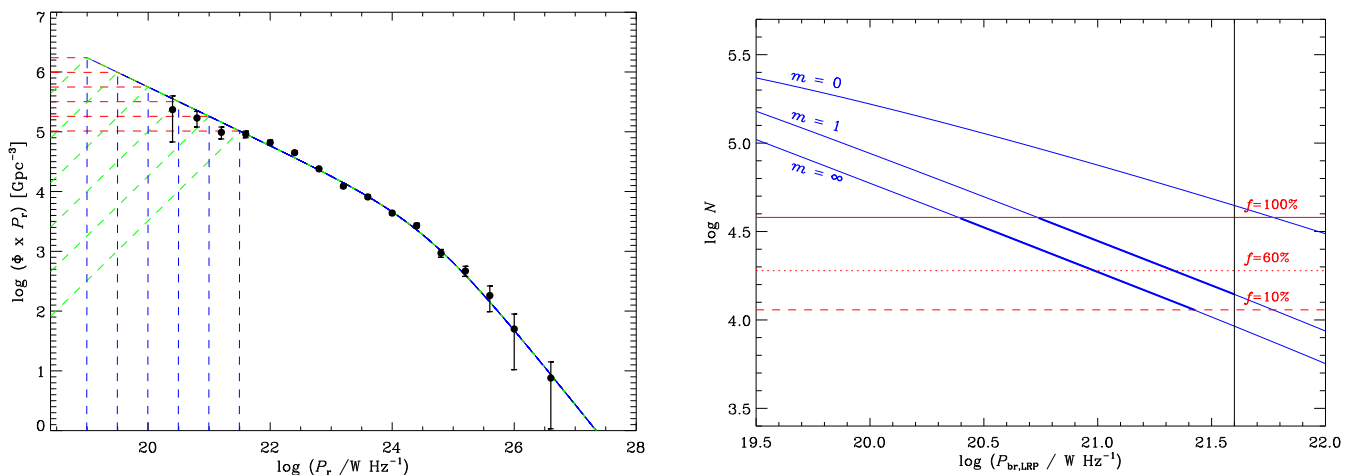
Throughout the Letter, we adopt  $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ,  $\Omega_M = 0.30$  and  $\Omega_\Lambda = 0.7$ .

## 2 THE RADIO AGN LUMINOSITY FUNCTION

The number density of AGN cannot exceed that of their potential hosts: this argument has been used by Mauch & Sadler (2007) and Cattaneo & Best (2009) to constrain the low-luminosity behaviour of their RLF. They found that a break must occur at a radio luminosity  $\log(P_r/\text{W Hz}^{-1}) \sim 19.5$  and 19.2, respectively. We use the same rationale, but including the information that their hosts, as shown in more detail below, are associated almost exclusively to very large black holes.

The RLF of nearby galaxies extending to the lowest luminosity has been obtained by Mauch & Sadler (2007). It is derived from a sample of 7824 radio sources from the 1.4 GHz NRAO VLA Sky Survey (NVSS) cross-correlated with the second data release of the 6 degree Field Galaxy Survey (Jones et al. 2005). Mauch & Sadler

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**Figure 1.** Left-hand panel: AGN number density as a function of the radio power  $P_r$  at 1.4 GHz. The results of Mauch & Sadler (2007) are shown as black dots and their fit as a solid blue line. The fit has been extrapolated below different values of the break luminosity,  $P_{\text{br,LRP}}$ , with power laws of different slopes  $m$  (red for  $m = 0$ , green for  $m = 1$ , and blue for  $m = \infty$ ). Right-hand panel: total number of predicted LRP AGN in the SDSS-DR7 area with  $0.05 \leq z \leq 0.1$  as a function of the low-luminosity break in the RLF, estimated for the three values of  $m$ . The solid horizontal line represents the total number of possible radio-galaxies hosts in the same volume; the other horizontal lines allow for smaller occupation fractions. The thick blue segments highlight the ranges of acceptable  $P_{\text{br,LRP}}$  values. The vertical line marks the upper limit to  $P_{\text{br,LRP}}$  set by the observations.

separated radio sources associated with AGN from star-forming galaxies relying on their optical spectra. Their resulting RLF covers the range  $\log(P_r / \text{W Hz}^{-1}) = 20.4\text{--}26.4$  and shows a steepening at high luminosities (above  $\log(P_r / \text{W Hz}^{-1}) = 24.59$ ).

Best & Heckman (2012) obtained an independent estimate of the AGN RLF. This is slightly shallower (reaching a luminosity  $\log(P_r / \text{W Hz}^{-1}) \sim 22$ ) but, since their sample is drawn from the Sloan Digital Sky Survey (SDSS), they could take advantage of the optical data to better characterize their properties. The main result is that their hosts are almost exclusively massive early-type galaxies with black hole masses (derived from the stellar velocity dispersion adopting the Tremaine et al. 2002 law) larger than  $\log(M_{\text{BH}}/M_{\odot}) \sim 7.8$  (see also Baldi & Capetti 2010).

The SDSS survey provides us with a highly complete sample that can be used to estimate the number density of galaxies with large black holes. To this purpose, we analyse the 818 333 galaxies (MPA-JHU sample hereafter) in the value-added spectroscopic catalogue produced by the Max Planck Institute for Astrophysics, and the Johns Hopkins University, and available at <http://www.mpa-garching.mpg.de/SDSS/> (Brinchmann et al. 2004; Tremonti et al. 2004). We consider all galaxies from redshift  $z = 0.05$  to 0.1 to ensure a high level of completeness (see below) within the largest possible volume, which results in  $0.053 \text{ Gpc}^3$ .

We then select elliptical galaxies in the MPA-JHU sample by setting a threshold to the concentration index  $C_r \geq 2.6$  (e.g. Strateva et al. 2001; Bell et al. 2003; Kauffmann et al. 2003) and to their stellar velocity dispersion adopting, conservatively,  $\sigma_{\text{star}} \geq 140 \text{ km s}^{-1}$ , corresponding to  $\log(M_{\text{BH}}/M_{\odot}) \geq 7.5$ , finding  $\sim 38\,000$  objects. The resulting sample has a high level of completeness. According to Montero-Dorta & Prada (2009), the completeness of the SDSS decreases with decreasing apparent magnitude, starting at  $\sim 95$  per cent at the SDSS spectroscopic limit of  $r = 17.77$ , and being still higher than 80 per cent at  $r = 13.25$ . The vast majority of the LRP AGN hosts have a magnitude in the range  $-23.5 < M_K < -26.5$  (Mauch & Sadler 2007). This translates into  $r = 13.3\text{--}16.3$  at  $z = 0.05$  (and  $r = 14.8\text{--}17.8$  at  $z = 0.1$ ) having adopted  $r - K = 3.0$  (Chang et al. 2006). This implies that most of such galaxies in the selected redshift range are included in the MPA-JHU catalogue.

We now estimate how the total number of predicted LRP AGN varies depending on the behaviour of the RLF at low power. A low-luminosity break must occur in order to avoid the divergence of the LRP AGN number. The Mauch & Sadler results suggest that it is located below  $\log(P_r / \text{W Hz}^{-1}) \sim 21.6$ , where the limited number of observed objects makes the RLF shape uncertain. We assume that the LRP AGN number density follows, below a break luminosity  $P_{\text{br,LRP}}$ , a power law with an index  $m$ . We investigate the effects of varying  $m$  from 0 (a flat number count distribution) to infinity (equivalent to a sharp cutoff below  $P_{\text{br,LRP}}$ ), see the left-hand panel of Fig. 1. We integrate numerically the RLF starting from  $\log(P_r / \text{W Hz}^{-1}) \sim 18.4$ , a factor of 100 below the observational limit: the total number of predicted LRP AGN grows rapidly at decreasing  $P_{\text{br,LRP}}$  (Fig. 1, right-hand panel). For  $m = 0$ , it exceeds the total number of available hosts when  $\log(P_{\text{br,LRP}} / \text{W Hz}^{-1}) \sim 21.8$ , a value that is inconsistent with the Mauch & Sadler results even assuming that *all* massive galaxies host an LRP AGN. For larger  $m$  (and already for  $m = 1$ ), the tension with the observations is eased, as we obtain  $\log(P_{\text{br,LRP}} / \text{W Hz}^{-1}) \sim 20.4\text{--}20.7$ . However, this requires an occupation fraction  $f$  of 100 per cent, i.e. all massive galaxies must harbour a radio-source with at least  $P_r \sim P_{\text{br,LRP}}$ . The bivariate RLFs, obtained by Mauch & Sadler splitting the sample into bins of infrared luminosity (their fig. 16), indicate instead values in the range  $f \sim 10\text{--}60$  per cent depending on the host magnitude. This implies that the values of  $P_{\text{br,LRP}}$  derived above should be considered as strict lower limits.

We conclude that: (1) the RLF of LRP AGN must break to a shallower slope for radio luminosities smaller than  $\log(P_r / \text{W Hz}^{-1}) \sim 20.5\text{--}21.5$ , and (2) the number density below  $P_{\text{br,LRP}}$  must significantly decrease, leading to a peak in its distribution.

### 3 THE BL LAC OBJECTS LUMINOSITY FUNCTION

In this section, we consider the RLF of BL Lacs at 1.4 GHz. We adopt the same line of reasoning followed above. In this case, the constraint on the RLF is derived imposing the requirement that

the total number of BL Lacs does not exceed the total number of appropriate (see below) radio sources in the same volume.

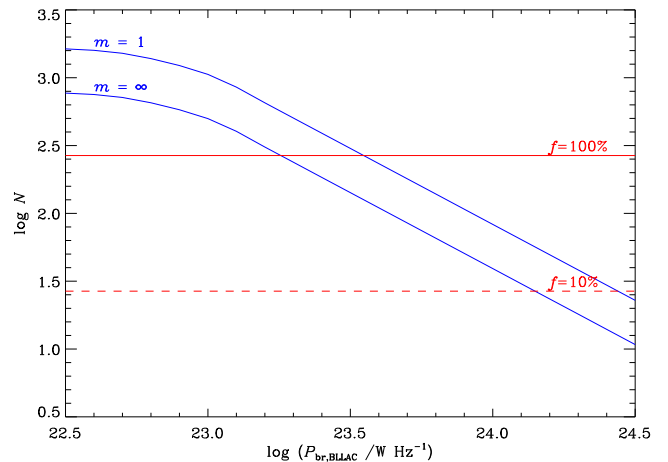
Best & Heckman (2012) provide us with a sample of 18 286 AGN obtained by combining the SDSS DR7 MPA-JHU sample with the NVSS and the Faint Images of the Radio Sky at Twenty-Centimeters (FIRST) surveys, down to a flux density level of 5 mJy in the NVSS. We consider the same volume as in Section 2, i.e.  $0.05 \leq z \leq 0.1$ . We select the sources with (1) a rest-frame equivalent width of all emission lines smaller than  $5 \text{ \AA}$  (the standard threshold for a BL Lac classification; Stickel et al. 1991), and (2) a flux of the central radio component in the FIRST images larger than 5 mJy. The latter requirement (based on the measurement of Best & Heckman 2012) is needed to isolate potential BL Lac objects according to their radio core emission while excluding, e.g. double-lobed sources lacking of a core. The threshold is set at the same radio flux limit that defines the selected sample and it is included self-consistently in the analysis that follows. We are left with 524 potential BL Lac objects. We further check their radio morphology: indeed BL Lacs are only associated with specific radio structures on the kpc scale, being dominated by a compact radio core and, in some cases, showing also a one-sided jet or a halo (Antonucci & Ulvestad 1985). We found that 49 per cent of the selected sources are instead well-defined Fanaroff–Riley type I or II (FR I or FR II) radio galaxies with sizes larger than 10–20 kpc. The 267 remaining objects are not necessarily all BL Lacs and certainly include intruders, for example, FR I with size smaller than  $\sim 10$  kpc that, due to the limited resolution of the FIRST images, are not recognized as such. However, with only a handful of exceptions, the selected objects are all massive early-type galaxies, conforming with the standard LRP AGN hosts.

On the other hand, there is the possibility that BL Lacs are so nucleary dominated to be confused with stars in the SDSS images (and for this reason they might not be included in the MPA-JHU galaxies catalogue). However, this possibility can be discarded. First of all, in the Roma-BZCAT catalogue of BL Lacs (Massaro et al. 2009) there are only eight objects with  $0.05 \leq z \leq 0.1$  and covered by the DR7. All of them have been selected as SDSS spectroscopic targets. Furthermore, the brightest of these objects has a radio luminosity of  $1.3 \times 10^{24} \text{ W Hz}^{-1}$ . We estimated the corresponding non-thermal contribution to the optical ( $r$ -band) magnitude for similarly bright objects by assuming, conservatively, a radio-to-optical spectral index of 0.5, typical of high energy peaked BL Lacs (Padovani et al. 2003). We obtain  $r = 17.5$  at  $z = 0.05$  (and  $r = 19.0$  at  $z = 0.1$ ),  $\sim 1$ – $4$  mag fainter than LRP AGN hosts. Only much brighter BL Lacs might be confused with stars. However, very luminous sources have a low space density and indeed no such object, that would have been easily discovered, is known within the volume considered.

We conclude that our estimate of 267 objects is a robust upper limit to the BL Lacs number in our volume.

Padovani et al. (2007) derived the RLF of BL Lacs at 5 GHz based on the Deep X-ray Radio Blazar Survey. They obtained an RLF consistent with a single power law with a slope of  $2.12 \pm 0.16$  in the range  $\log(P_r/\text{W Hz}^{-1}) \sim 23.8$ – $26.8$ . By adopting this RLF, we estimate the predicted number of BL Lacs in the selected volume when varying the luminosity,  $P_{\text{br, BLLAC}}$ , at which the RLF breaks, see Fig. 2. In our calculation, we assume a null radio spectral index<sup>1</sup> to convert from 5 to 1.4 GHz and take into ac-

<sup>1</sup> This assumption is based on the average radio spectral index of BL Lacs with  $z < 0.1$ , i.e.  $\langle \alpha_r \rangle = 0.03$  (Stickel et al. 1991). A steeper radio spectrum would slightly increase the derived limits on  $P_{\text{br, BLLAC}}$ .



**Figure 2.** Total number of predicted BL Lacs in the range  $0.05 \leq z \leq 0.1$  in the SDSS/DR7 area as a function of the low-luminosity break of their RLF,  $P_{\text{br, BLLAC}}$ , and of the slope  $m$  of the number density below the break. The solid horizontal line represents the total number of radio galaxies (weak-lined and core-dominated, see text for details) in the same volume; the dashed horizontal line corresponds to a BL Lacs percentage of the selected radio-sources of 10 per cent.

count that objects with radio flux density smaller than 5 mJy are not included in our sample (this effect causes the flattening of  $\log N$  for  $\log(P_{\text{br, BLLAC}}/\text{W Hz}^{-1}) \lesssim 23$  in Fig. 2). We conclude that the number of BL Lacs predicted by extrapolating their RLF to low luminosities exceeds the total number of possible hosts for  $\log(P_{\text{br, BLLAC}}/\text{W Hz}^{-1}) \lesssim 23.3$ – $23.5$ , depending on the RLF slope below the break. On the other hand, the RLF data points of Padovani et al. exclude  $\log(P_{\text{br, BLLAC}}/\text{W Hz}^{-1}) \gtrsim 24.5$ .

We conclude that the RLF of BL Lacs must break to a shallower slope for radio luminosities smaller than  $\log(P_r/\text{W Hz}^{-1}) \sim 23.3$ – $24.5$ .

## 4 DISCUSSION AND CONCLUSIONS

The results presented indicate that the RLFs at 1.4 GHz of both LRP AGN and BL Lacs must decrease their slope at, or just above, the low-luminosity limit at which they are currently determined. This might appear as a contrived coincidence, while, conversely, it is the natural consequence of the behaviour of the RLF at low power. In fact, the number of objects below the break that can be detected in a flux-limited survey depends on the radio power as  $P_r^{3/2+m}$ , where  $m$  is the number density slope for  $P_r < P_{\text{br}}$ . The strong dependence on  $P_r$  implies that these objects are missing altogether in the currently available surveys (or at most found in a small number) and this prevents the RLF determination below  $P_{\text{br}}$ .

For LRP AGN, we find that the number density below the break must significantly decrease producing a peak in the distribution. It is tempting to associate the presence of this peak to a minimal level of emission that is invariably associated with a supermassive black hole located at the centre of a massive galaxy, which unavoidably accretes gas from the interstellar medium. The peak might instead correspond to a minimum level of accretion at which a radio jet can be produced. A better definition of the RLF at low luminosity is essential in this context: in the first case, we expect to find that all potential hosts harbour an LRP AGN, albeit of very low luminosity while, in the second alternative, the occupation fraction would never reach 100 per cent. Note that

Kimball et al. (2011) found a similar dramatic fall of the number density from the analysis of a sample of 179 quasi stellar objects (QSOs) ( $M_i < -23$ ) from the SDSS in the redshift range  $0.2 < z < 0.3$ , for radio luminosities below  $\log(P_r/\text{W Hz}^{-1}) \sim 21$  (at 6 GHz).

Unfortunately, we have only sparse information on the radio properties of massive galaxies below  $\log(P_r/\text{W Hz}^{-1}) \sim 20$ . These can be obtained with deep targeted observations of nearby galaxies. In the Virgo clusters there are two (out of 11) giant elliptical galaxies lacking of any sign of radio emission down to a limit of  $\log(P_r/\text{W Hz}^{-1}) \sim 18.6$  (Capetti et al. 2009), despite their large mass ( $M_* \sim 10^{11.5} M_\odot$ ) and large black hole mass ( $M_{\text{BH}} \sim 10^8\text{--}10^{8.5} M_\odot$ ). The small number statistics prevent us to derive any firm conclusion based on these observations, but they apparently favour the interpretation that not all massive galaxies are able to produce a radio jet.

The two RLFs considered in this Letter are expected to be linked with each other. Indeed, according to the AGN unified scheme, BL Lacs are the beamed version of low-luminosity AGN (see e.g. Urry & Padovani 1995; Tadhunter 2008), i.e. BL Lacs are objects in which a highly relativistic jet is seen at an angle close to our line of sight. This accounts for their extreme properties, such as strong flux variability and apparent superluminal motion of radio components. The effects of relativistic beaming on the RLF was analysed in a series of papers (Urry & Shafer 1984; Urry, Padovani & Stickel 1991; Urry & Padovani 1991, 1995). If the RLF of the parent population of unbeamed objects is described by a broken power law, the break will produce a change of slope in the BL Lac RLF at  $\sim \delta_{\text{max}}^3 \times P_{\text{br,LRP}}$ , where  $\delta_{\text{max}}$  is the maximum value of the jet Doppler factor distribution.<sup>2</sup> The ratio between  $P_{\text{br,LRP}}$  and  $P_{\text{br,BLLAC}}$  is a factor of  $\sim 10^3$ , which suggests a value of the Doppler factor  $\delta_{\text{max}} \sim 10$ , remarkably consistent with the typical results obtained from the observations and the model predictions (e.g. Ghisellini et al. 1993; Savolainen et al. 2010). This analysis implicitly assumes that all LRP AGN produce relativistic jets, a requirement that is not necessarily met by the objects at the very faint end of the RLF. The consistency between the estimates of the Doppler factor we just obtained with the theoretical expectations appears to confirm the overall validity of this assumption.

How can we improve our knowledge of the RLF at low power? For LRP AGN the lower limit to the break power,  $\log(P_{\text{br,LRP}}/\text{W Hz}^{-1}) \sim 20.5$ , is not far from what is currently accessible to radio observations of large sample of galaxies such as the NVSS. A factor of  $\sim 10$  improvement in the flux threshold is already accessible to the Karl G. Jansky Very Large Array (JVLA) and it will be sufficient to measure directly the RLF of LRP AGN below the break value we predicted. While waiting for such a survey it is possible to take advantage of the stacking technique (e.g. White et al. 2007) to measure the median radio luminosity of the population of massive elliptical galaxies at sub-mJy levels. In case, the number density presents a peak in its distribution, as our results suggest, this median luminosity is predicted to be located close to the peak value.

<sup>2</sup> The Doppler factor  $\delta$  is defined as  $\delta = [\Gamma(1 - \beta \cos \theta)]^{-1}$ , where  $\Gamma = (1 - \beta^2)^{-1/2}$  is the bulk Lorentz factor,  $\beta$  is the velocity in units of the speed of light and  $\theta$  is the viewing angle. The exact value of the  $\delta$  exponent depends on the geometry of the emitting region.

## ACKNOWLEDGEMENTS

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